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Criteria for degree of observability in a control system

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IN A CONTROL SYSTEM

by

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I. INTRODUCTION

A paper, presented in 1960 by R. E. Kalman (18) at the First International Congress on Automatic Control in Moscow, U. S. S. R., suggested the use of vectors and matrices to analyze control systems and introduced the concepts of controllability and observability. This paper was among the first of many papers in a new area of system theory called state-variable theory. This theory yields a more fundamental understanding of the system than the transfer function approach previously used.

With this theory a group of new terms have been introduced. The first of these terms, state of a dynamic system is defined as the smallest collection of numbers which must be specified at a present time, t_o, in order to be able to predict the future behavior of the system, provided the system's mathematical formulation and future inputs are known.

The state-variables of a dynamic system are the elements of the states as the elements vary with time. These state-variables represent the physical quantities or a linear combination of the physical quantities internal to the system.

The state-variable formulation can be compared to the transfer function approach which deals entirely with input and output quantities of the system. A large system may contain some modes of operation over which the input may have no control or which may never appear in the output. These modes of operation would never appear in the transfer function approach. The concepts of controllability and observability deal with these "missing modes of operation" and will be discussed later.

State-variable theory gives a much more complete mathematical description of a dynamic system and is able to accommodate systems with multi-inputs and multi-outputs much better than the transfer function approach. In addition, the transfer function approach can be said to be a subset of the state-variable theory, because the transfer function can always be derived from the state-variable description of the system, but the reverse is not always true. DeRusso, Roy, and Close (11) states, "From a mathematical viewpoint, the state-variable approach is the use of matrix and vector methods to handle the large number of variables which enter into such problems. As such, these are not new methods, but rather they are the rediscovery of existing mathematical techniques. They aid considerably in the solution of linear multivariable problems. More important, however, the state-variable approach aids conceptual thinking about these problems . . . ."

Since this thesis is concerned with linear dynamic systems, all the following discussion will be restricted to the linear dynamic systems.

A. State-Variable Formulation

The mathematical formulation of a linear dynamic system in state-variable theory is forced to fit the following two matrix equations.

$$\dot{x} = Ax + Bv \tag{1}$$

$$y = Cx \tag{2}$$

where

x = n x 1 column vector of the state-variables.

$\dot{x}$ = n x 1 column vector of the time derivatives of the state variables.

A = n x n matrix giving the relation between x and $\dot{x}$.

v = p x 1 column vector of the inputs variables to the system.

B = n x p matrix coupling the inputs variables to the system.

y = m x 1 column vector of the output variables of the system.

C = m x n matrix coupling the state-variables to the output variables.

If at some time, t_o, the state of the system, $x(t_o)$, is known, these matrix equations can be solved to give the following equation.

$$x(t_1) = \varphi(t_1, t_o)\, x(t_o) + \int_{t_o}^{t_1} \varphi(t_1, \tau)\, B(\tau)\, v(\tau)\, d\tau \qquad (3)$$

The matrix, $\varphi(t_1, t_o)$, is called the transition matrix. It is the solution of Equation 1 when the input vector, v, is zero. As can be seen in Equation 3, when the input vector, $v(\tau)$, is zero, the transition matrix would relate the state of the system at time, t_o, to the state of the system at time, t_1. More information on state-variable theory can be found in DeRusso, Roy, and Close (11), or Zadeh and Desoer (33), or many other books or papers written about the subject.

B. Observability and Controllability

The definition for observability given in a paper by Kalman (18) was later modified by Gilbert (13) and accepted by Kalman (17). The following definitions found in Zadeh and Desoer (33) agree with Gilbert's definition and are fairly well accepted.

Controllability

A system is said to be controllable if and only if for any state, there is an input which will reduce the state to zero in a finite time. If all states are controllable, the system is said to be "completely controllable".

Observability

A system is said to be observable if and only if in some finite time after t_o with the knowledge of the state-variable description of the system and with zero inputs, the initial state at time, t_o, can be determined by observing the output variables.

The preceding definition for controllability and observability gives good physical insight into the concept of each, but does not aid much in determining the controllability or observability of a system from the mathematical point of view. For this reason, some authors prefer to define controllability and observability on the basis of a Q matrix. Brown (7), in a paper presented at the National Electronics Conference in 1966, has a very good discussion showing that the Q matrix criterion is derived from the basic definition of observability given above for both the time-invariant and time variable systems.

For the time-invariant system, the Q matrix is formed as shown below for both controllability and observability.

Controllability Q matrix:

$$Q = [B, AB, A^2B, \dots A^{n-1}B] \tag{4}$$

Observability Q matrix:

$$Q = [C^T, A^TC^T, (A^T)^2 C^T, \dots (A^T)^{n-1} C^T] \tag{5}$$

The superscript T means the transpose of the matrix and n is the order of the A matrix. The criterion for a controllable or observable system is that there be n independent columns in the Q matrix. This criterion can also be stated as the rank of the Q matrix must be equal to n. For the time-invariant system, Chen, Desoer, and Niederlinski (9) has shwon that the complete Q matrix may not be needed to determine its rank. They determine the rank for the B or C^T part of the matrix first, i.e., the first p or m columns where p refers to the controllability Q matrix and m refers to the observability Q matrix. The symbols p and m are defined in Equations 1 and 2 and are respectively the number of inputs and outputs of the system. They then add the next p or m columns to the part already checked and determine its rank. They keep adding p or m columns until the ranks of two successive matrices are equal. The last rank determined is the rank of the complete Q matrix.

For time variable systems, i.e., where the matrices A, B, or C may be functions of time, the Q matrix formulation is more complicated. This development can be found in at least two places in the literature. The paper by Brown (7) has one development. Silverman and Meadows (31) gives another development. Only the results of the development are given here. The notation used here is somewhat similar to that used by Silverman and Meadows (31). A sequence of matrices, P_1, P_2, ··· P_k, ··· P_n is defined where n is the order of the A matrix as defined in Equations 1 and 2. The sequence is defined as shown by the set of Equations 6 and 7.

Controllability: $P_1 = B$

$$\vdots \quad \vdots$$

$$P_k = A \; P_{k-1} + \frac{d}{dt} P_{k-1}$$

$$\vdots \quad \vdots$$

$$\text{Observability:} \quad \begin{aligned} P_1 &= C^T \\ &\vdots \\ P_k &= A^T \; P_{k-1} + \frac{d}{dt} P_{k-1} \\ &\vdots \end{aligned} \tag{7}$$

The Q matrix is defined as shown in Equation 8.

$$Q = [P_1, P_2 \cdots, P_k, \cdots P_n] \tag{8}$$

The criterion on the Q matrix is the same here as before, namely, that there be n independent columns for a controllable or observable system. It should be noted that this definition and criterion will also work for the time-invariant system.

The paper by Silverman and Meadows (31) also shows that any Q matrix composed of more than n matrices from the sequence will have the same rank as a Q matrix composed of only n matrices of the sequence.

Another criterion for controllability and observability has been developed using the transition matrix instead of the A matrix. Since this thesis is based on the Q matrix no further discussion on the criterion will be given here; however, more information may be found in a paper by Kreindler and Sarachik (19).

Since observability is the main subject to be considered in this thesis, the rest of the discussion will concentrate on observability with controllability being left to follow by analogy.

II. NOT JUST OBSERVABLE, BUT HOW OBSERVABLE

All the criteria presently available for observability give a "yes-no" answer with no indication as to how close to the dividing line the system may be. Brown (7, 8) has opened the issue of "How Observable?" In the development of the observability Q matrix, Brown points out that this matrix relates the state-variables to the output variables and the derivatives of the output variables. It is done in the following manner for the time-invariant system. Starting with Equations 1 and 2, assuming the input to be zero, Equation 2 is differentiated and Equation 1 is substituted as shown below.

$$\begin{aligned}
y(t_o) &= Cx(t_o) \\
\dot{y}(t_o) &= C\dot{x}(t_o) = CAx(t_o) \\
\ddot{y}(t_o) &= CA\dot{x}(t) = CA^2x(t_o) \\
&\vdots \\
y^{n-1}(t_o) &= CA^{n-1}x(t_o)
\end{aligned} \tag{9}$$

Equation 2 is differentiated n-1 times because the theorem due to Silverman and Meadows (31) shows that any further differentiation is superfluous.

The set of equations numbered 9 can be rewritten in the matrix form shown by Equation 10.

$$\begin{bmatrix} y(t_o) \\ \dot{y}(t_o) \\ \ddot{y}(t_o) \\ \vdots \\ y^{n-1}(t_o) \end{bmatrix} \begin{matrix} = \\ = \\ = \\ \\ = \end{matrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} [x(t_o)] \tag{10}$$

Let the column vector on the left of Equation 10 be $y_d(t_o)$. By inspection the matrix to the right of the equal sign in Equation 10 can be seen to be Q^T.

The matrix Equation 10 may be rewritten as shown by Equation 11.

$$y_d(t_o) = Q^T x(t_o) \tag{11}$$

Brown (6), shows a similar development for the time variable system. The results are the same as shown by Equation 7 and Equation 11.

If the system has only one output, the state, $x(t_o)$, can be found by inverting the Q^T matrix. However, the inverse of Q^T only exists if the determinant of Q^T is nonzero or, in other words, if the rank of the matrix is equal to its order. If the system has multiple outputs it should be possible to pick n linearly independent columns of the Q matrix and invert the square matrix. However, the criterion that the Q matrix have n independent column also means that the rank be n. Thus, it is now clear from where the "yes-no" answer to the observability question came.

Brown proposes that the degree of independency of the columns of the Q matrix is also the degree of observability of the system. For example, if n columns of the Q matrix are orthogonal, the degree of independency of the columns is as high as possible, and the system will be highly observable. If a vector can be found which is nearly orthogonal to all the columns of the Q matrix, then the degree of independency of the columns would be low; likewise the degree of observability for the system would be low. In this last case difficulty would be encountered in solving Equation 11. A small measurement error would be reflected as a large error in the solution of the unknowns.

Furthermore, the direction of the "nearly orthogonal vector" indicates the direction of greatest error in the solution of the state-variables. If, for example, a three state-variable system had the "nearly orthogonal vector" pointed half way between state variable number 2 and 3, they would have the greatest error while state-variable number 1 would have the smallest error, if all the observation errors were equal. These equations are known as ill-conditioned and further discussion can be found in a paper by Gavurin (12).

Since the "most orthogonal" vector conveys considerable information, the next problem to be discussed is the evaluation of it. The development shown here is due to Brown (6) in his unpublished notes. First, the columns vectors of the Q matrix must be normalized because we are more interested in the "angles" between the columns vectors and "most orthogonal" vector rather than the "length" of the vectors. The Q matrix with its columns normalized will be designated as Q_N and its columns as w_1, w_2, w_3, $\cdots$ w_{mn}. Brown forms an observability function called L which is a scalar as shown by Equation 12.

$$L = (w_1{}^T u)^2 + (w_2{}^T u)^2 + \cdots (w_{mn}{}^T u)^2 \qquad (12)$$

The symbol, u, is the "most orthogonal" vector with the constraint that it be of unit length.

Equation 12 may be rewritten as shown in Equation 13.

$$L = u^T[w_1w_1^T + w_2w_2^T + \cdots + w_{mn}w_{mn}^T]\ u \tag{13}$$

By expansion of the $Q_NQ_N^T$ matrix, it can be shown that the $Q_NQ_N^T$ matrix is the quantity inside the brackets of Equation 13. Equation 13 may be rewritten as Equation 14.

$$L = u^T(Q_NQ_N^T)\ u \tag{14}$$

This problem is a maxima-minima type problem very suitable to the method of Lagrangian multipliers as given in Chapter 4, Section 5 of Widder (32). In this case the constraint is expressed by Equation 15 and declares that the "most orthogonal" vector must be of unit length.

$$u^Tu = 1 \tag{15}$$

The Lagrangian multiplier formulation is given by Equation 16 where λ, a scalar, is the Lagrangian multiplier.

$$\frac{d}{du}\ [u^T(Q_NQ_N^T)\ u - \lambda\ (u^Tu-1)] = 0 \tag{16}$$

The indicated differentiation is of quadratic form. More details on it can be found on pages 288-289 in DeRusso, Roy, and Close (11). The result of the differentiation is given by Equation 17.

$$(Q_NQ_N^T - \lambda I)u = 0 \tag{17}$$

The matrix I is the unit matrix. From Equation 17, it is clear that the "most orthogonal" vector is an eigenvector of the $Q_NQ_N^T$ matrix. To determine the correct eigenvector, the eigenvectors can be substituted into Equation 14. The eigenvector which yields the smallest value of L is the "most orthogonal" vector. Another way of determining the correct eigenvector requires the following development. Rearrange Equation 17 and premultiply both sides by u^T. The result is as shown by Equation 18.

$$u^T(Q_N Q_N{}^T)u = u^T \lambda u = \lambda = L \tag{18}$$

This equation shows that the observability function is equal to the smallest eigenvalue of the $Q_N Q_N{}^T$ matrix. Therefore, the "most orthogonal" vector is the eigenvector associated with the smallest eigenvalue.

Since the smallest eigenvalue is the observability function, its value gives a measure of the system observability. A small value of the observability means that one or more elements of a state will have a large error associated with it when determined from observations which has measurement error. All the eigenvalues being equal means that all the elements of a state are as observable as they can be.

In order to gain an idea of what the values of the observability function mean, a theorem due to Bocher as expressed on page 234 in DeRusso, Roy, and Close (11) will be used. The theorem states that the sum of all the eigenvalues of a matrix is equal to the trace of the matrix. An expansion of the trace of $Q_N Q_N{}^T$ shows that the trace is always equal to the number of non-zero columns of the Q matrix. Since all the eigenvalues of a "most observable" system are equal, the value of the observability function of a "most observable" system is equal to the number of non-zero columns of the Q matrix divided by n. In the case, where there are no non-zero columns, the value is equal to the number of outputs of the system.

When the smallest eigenvalue is zero, the system is unobservable. If any of the state-variable have a component in the same direction as the eigenvector associated with the zero eigenvalue, that state variable is unobservable. All the other state variables are observable. However, by the definition of observability given earlier the system is still unobservable.

The preceding procedure is very useful for a small system, but when systems get larger and more complex, it sometimes becomes necessary to consider the second or third "most orthogonal" vector. For these cases, the procedure described in the next section should be helpful.

III. DEGREES OF OBSERVABILITY PER STATE-VARIABLE FOR THE SINGLE-OUTPUT OBSERVABLE SYSTEM

For systems which are nearly unobservable, we are interested in which state variables are most observable and which ones are least observable. We are interested in finding a figure of merit for each state-variable which will reflect how observable the state-variable is. The criterion selected for this thesis is based on the increase in error of the calculated state-variable over the error in the observations.

With Kalman filter theory as explained in Lee (20) the error in the calculated state variable can be found. However, the work involved is much greater than the method proposed here; and, the error in each observation must be known and specified. In the method proposed here, the error in each observation is assumed to be equal to the error in all the other observations in a "pseudo-normalized sense". Thus, the method presented here yields a relatively quick and easy means of gaining some insight into the degree of observability without going through the entire Kalman estimation procedure.

In defining the <u>Degree of Observability per State-Variable</u> we will use the reciprocal of the increase in the error of the calculated state-variable over the observation error. The reciprocal is used so that a small number will result for nearly unobservable state-variables.

The two most common approaches to error analysis is the upper-bound error and the standard-deviation error. The criterion has been developed for both approaches.

Let us develop the criterion for the single-output observable system first, and consider the multi-output and unobservable systems later.

Referring to matrix Equation 11, we will normalize the rows of the Q^T matrix and divide the elements of the $y_d(t_o)$ vector by the length of the corresponding row vectors of the Q^T matrix. We will define the normalized vector as $y_{dN}(t_o)$ and the normalized matrix as $Q_N{}^T$. The equation may be written as Equation 19.

$$y_{dN}(t_o) = Q_N{}^T[x(t_o)] \tag{19}$$

The vector $y_{dN}(t_o)$ consists of the actual value of the vector and an error term and can be split into the two vectors, the actual value, $y_a(t_o)$ and error, e. Equation 19 can then be rewritten as Equation 20.

$$[y_a(t_o)] + [e] = Q_N{}^T[x(t_o)] \tag{20}$$

Solving Equation 20 by taking the inverse of $Q_N{}^T$ results in Equation 21.

$$x(t_o) = (Q_N{}^T)^{-1}\ [y_a(t_o)] + (Q_N{}^T)^{-1}\ [e] \tag{21}$$

Two items should be noted. First, the matrix $Q_N{}^T$ is square and invertible because we are considering only a single output observable system. Second, the elements of the e vector are not the actual measurement errors of the observations, but are modified by being divided by the length of the corresponding row vectors of the Q^T matrix.

Equation 21 shows that the calculated state of the system is split into the actual state plus the error of the calculated state. The equation shows that this error is the linear combinations of the measurements errors.

Let us consider the upper-bound error first. We will replace the elements of the e vector with the "modified upper-bound errors" for each measurement putting the usual plus or minus sign in front of each vector element. Since we are looking for the upper-bound error on the calculated value of each state-variable, we must select the signs of the elements in the e vector to yield the maximum calculated error. Since the calculated error is a linear combination of the observation errors, the calculated error turns out to be the sum of the absolute values of the row coefficients of $(Q_N^T)^{-1}$ when each coefficient is multiplied by its respective observation error. If we let the elements of the e vector be equal, we see these elements will cancel when the ratio for the degree of observability is calculated. We are left with a single number which is our degree of observability per state-variable for the upper-bound error. To shorten this name we will call it upper-bound observability. To recapitulate, the upper-bound observability for a state-variable is the inverse of the sum of the absolute values of the coefficients in the corresponding row in the $(Q_N^T)^{-1}$ matrix.

To find the corresponding degree of observability per state-variable when standard deviation is used as a measure of error, we will refer to a theorem from statistics found on page 126 of Lindley (21). The theorem states that the variance of the linear combination of independent random variables is the sum of the coefficients squared multiplied by the respective variances of each random variable. The standard deviation is then the square root of the variance. Applying this theorem to our case, we know that the calculated error is a linear combination of the measurement errors. Therefore, the calculated error is the sum of the coefficients squared in the row of the $(Q_N^T)^{-1}$ multiplied by the variance of each measurement. An expansion of $(Q_N^T)^{-1}$ will show this condition. Again when the ratio is taken to find the degree of observability per state-variable, we find that, if the variances in the e vector were all made equal, they would cancel. Therefore, the degree of observability per state-variable based on the standard deviation is then the reciprocal of the square root of the sum of the coefficient squared in the respectively rows of the $(Q_N^T)^{-1}$ matrix. To shorten the name we will call it standard-deviation observability.

At this point a simple example will be given to make the preceding discussion clearer. Consider the following system. (Figure 1).

The state-variable formulation is given by Equations 22 and 23.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -12 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{22}$$

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{23}$$

Step 1: Form the Q matrix as specified by Equation 5.

$$Q = \begin{bmatrix} 0 & 3 \\ 1 & -4 \end{bmatrix} \tag{24}$$

Step 2: Normalize the columns of the Q matrix.

$$Q_N = \begin{bmatrix} 0 & 0.6 \\ 1 & -0.8 \end{bmatrix} \tag{25}$$

Step 3: Form the $Q_N Q_N^T$ matrix.

$$Q_N Q_N^T = \begin{bmatrix} .36 & -.48 \\ -.48 & 1.64 \end{bmatrix} \tag{26}$$

Step 4: Find the eigenvalues and the corresponding eigenvectors.

(For procedure, see Ralston (28) Chapter 10, pages 487-499.)

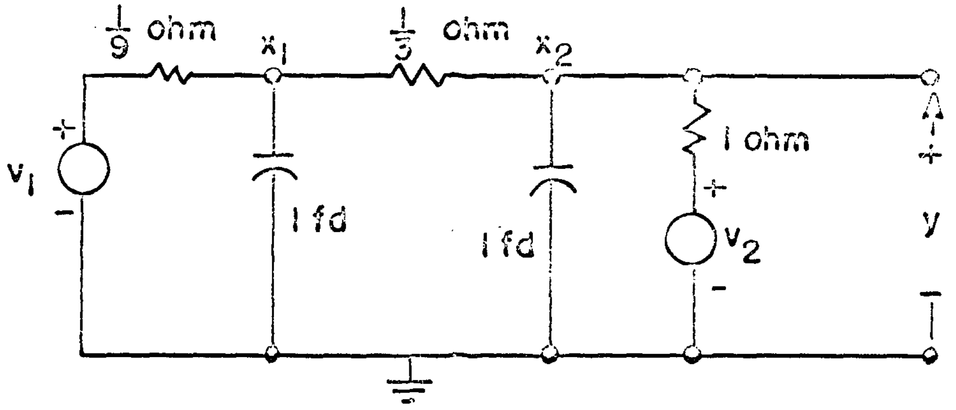

Figure 1. Circuit for simple example

$$\lambda_1 = 0.2; \; u_1 = \begin{vmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{vmatrix} \tag{27}$$

$$\lambda_2 = 1.8; \; u_2 = \begin{vmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{vmatrix} \tag{28}$$

The observability function is the smallest eigenvalue which has a value of 0.2. Since there are only two columns in the Q matrix, the sum of the eigenvalues is two. Since the "most observable" system would have both eigenvalues equal to one, we can gain an idea of the observability of this system by comparing the observability function value of 0.2 to the value of one. By examining the eigenvector associated with the 0.2 eigenvalue, we see that the state-variable, x_1, is less observable than the state-variable, x_2. This result is very satisfying because x_2 is measured directly while x_1 has to be calculated.

Proceeding with the rest of the example.

Step 5: Find the inverse of Q_N^T.

$$(Q_N^T)^{-1} = \begin{vmatrix} 4/3 & 5/3 \\ 1 & 0 \end{vmatrix}$$

Step 6: Form the degree of observability per state-variable for each type of error.

Upper-bound observability:

For state variable number 1

$$\frac{1}{|4/3| + |5/3|} = \frac{1}{3} = 0.333$$

For state variable number 2

$$\frac{1}{|1| + |0|} = \frac{1}{1} = 1$$

Standard-deviation observability:

For state variable number 1

$$\frac{1}{(4/3)^2 + (5/3)^2} = \frac{1}{2.134} = 0.4685$$

For state variable number 2

$$\frac{1}{(1)^2 + (0)^2} = \frac{1}{1} = 1$$

The upper-bound observability for x_1 says that the upper-bound error for $x_1(t_o)$ will be three times the upper-bound error on the measurements of $y(t_o)$ and $\dot{y}(t_o)$ if the modified upper-bound error for each measurement were equal.

The upper-bound error for $x_2(t_o)$ will be the same as the measurement upper-bound error. Again, the result is very satisfying because $x_2(t_o)$ is measured directly.

By analogy, the standard-deviation observability for $x_1(t_o)$ shows that the standard deviation for $x_1(t_o)$ is 2.134 times the standard deviation on the measurements. Likewise, the standard deviation of $x_2(t_o)$ is the same as the measurement standard deviation.

This simple example does not show the advantage of the degree of observability per state-variable because the engineer can essentially gain all the needed information from the "most orthogonal" vector. However, later examples will be given where more than the "most orthogonal" vector will be helpful.

IV. DEGREES OF OBSERVABILITY PER STATE-VARIABLE FOR THE GENERAL CASE

We have yet to consider the systems with multiple outputs and systems which are not observable. In the case of the non-observable systems, we are interested in how observable the state-variables are which can be observed. For the multi-output system the Q matrix is not square. In the preceding discussion, we took the inverse of the Q_N^T matrix; however, for both cases presented above the simple inverse of the Q_N^T matrix cannot be found.

The answer to the above problem is the generalized inverse (frequently called peusdo-inverse). E. H. Moore (22) discovered the generalized inverse in 1920. It was rediscovered independently by A. Bjerhammar (3, 4) in 1951 and by R. Penrose (25) in 1955. T. N. E. Greville (15, 16), in papers published in 1959 and 1960, gives the information about the history of the generalized inverse.

Only the essential features of the generalized inverse will be given here. Besides the papers already mentioned, further information may be obtained by referring to any of the following papers (1, 2, 10, 14, 23, 24, 26, 27, 29, and 30).

Consider the matrix Equation 29.

$$Tz = b \tag{29}$$

Let us assume first there are more rows in T than in z but with the rank of T equal to the number of elements in z. In this case, there is the possibility of conflicting data in the b vector. The generalized inverse, written as T^+, would yield a vector z_o which would be the best fit to the data in the least squares sense. The vector z_o is specified as given in Equation 30.

$$z_o = T^+ b \tag{30}$$

The best fit in the least squares sense is specified by Equation 31.

$$|| \; Tz_o - b \; || \le || \; Tz - b \; ||; \text{ for any } z \tag{31}$$

The double lines denote the commonly defined length of the vector.

If the rank of T is less than the number of elements in the vector z or if there are fewer rows in T than elements in z, there are many vectors, z, which will fit Equation 31. For this case the generalized inverse will yield the z_o whose length is shorter than all other z which will fit Equation 31. This condition is described by Equation 32.

$$|| \; z_o \; || \le || \; z \; ||; \text{ for all } z \tag{32}$$

To be more precise mathematically, the generalized inverse is frequently defined by Penrose's (25) four equations given by Equations 33, 34, 35, 36.

$$TT^+T = T \tag{33}$$

$$T^+TT^+ = T^+ \tag{34}$$

$$(TT^+)^H = TT^H \tag*{(33)}$$

$$(T^+\bar{T})^H = T^+T \tag*{(34)}$$

The superscript H stands for the hermitian of the matrix and indicates that the matrix with the superscript is the complex conjugate transpose of the matrix without the superscript. Penrose has shown that these four conditions will always define a unique generalized inverse.

Zadeh and Desoer (33) has an interesting diagram on page 578 of their book which points out an interesting property of the generalized inverse concerning its null space. The null space of a matrix is defined as the set of all vectors z such that the product of the matrix, T, times the vector z is equal to zero. The diagram shows that the generalized inverse will never transform anything into the null space of the original matrix.

Zadeh and Desoer (33) also presents a method of finding the generalized inverse of pages 581-582. This was the method used in the computer program implementing these techniques because part of it is similar to the work which has to be done to find the observability function and "most orthogonal vector".

The method is as follows. Let the matrix S be the hermitian non-negative definite matrix defined by Equation 35.

$$S = T^H T \tag{35}$$

Let U be the matrix whose columns are the normalized eigenvectors of S so that the diagonal matrix D of the eigenvalues results when the similarity transformation given by Equation 36 is performed.

$$D = U^{-1} S U \tag{36}$$

In this case, since U is an orthonormal matrix, the hermitian of it is equal to its inverse.

The generalized inverse of the diagonal matrix D is the diagonal matrix D^+ whose diagonal elements are the reciprocal of the corresponding elements in the D matrix. If a diagonal element in D is zero it is left at zero in the D^+ matrix. The generalized inverse of the matrix T is given by:

$$T^+ = U D^+ U^{-1} T^H \tag{37}$$

It should be noted here that for real matrices the hermitian of the matrix is equal to the transpose of the matrix.

The last property to note about the generalized inverse is the fact that it becomes the inverse of the matrix when the matrix is square and non-singular.

With all the properties that the generalized inverse possesses, it fits very well into the scheme of things for the multi-output and unobservable system. We will always take the generalized inverse of Q_N^T in place of the inverse and proceed as described in the preceding section for calculating the degree of observability per state-variable.

V. IMPLEMENTATION ON THE COMPUTER

A computer program was written to calculate the observability function, "most orthogonal vector", upper-bound observability, and standard-deviation observability. Two linear systems for which the results were known were checked with the criteria developed in this thesis. The program was written in BPS Fortran and runs were made on the IBM 360 Model 50 computer in use at Iowa State in the Fall of 1966.

The Fortran program is given in Appendix A. The program is quite straight forward and follows the preceding development. The program used for the calculation of the eigenvalues and eigenvectors of $Q_\Lambda Q_\Lambda^T$ is due to the method by Jacobi found in Ralston (28). It is a slight modification of the program from the computing system library. The subroutine Fortran program is given in Appendix A. One of the disadvantages of this method is that the zero eigenvalues do not come out to be identically zero but are left at some small number. Therefore, a threshold has to be calculated to determine when the eigenvalue should be zero.

The generalized inverse is calculated as discussed in Chapter IV of this thesis by the method given in Zadeh and Desoer (33). To check on the accuracy of this method, a method of calculating the generalized inverse given by Rust, Burrus, and Schneeberger (30) was programmed. The method due to Zadeh and Desoer gave poor accuracy until the double precision feature of the computing system was employed. The modified Fortran program due to Rust, Burrus, and Schneeberger is given in Appendix B.

VI. DISCUSSION OF RESULTS

Variations of two different systems were used to calculate observability functions and degrees of observability. The first is an inertial navigation system due to Bona (5). The second is an inertial navigation system due to Brown (8).

The A and C matrices of the system due to Bona are presented in Table 1. The values of the numbers are presented in Table 3. This system was first checked with the last three state-variables eliminated; and finally, with all nine state-variables present. The results are shown respectively in Computer Output Number 1 and 2. All eigenvalues of the $Q_N Q_N^T$ matrix are presented in Computer Output Number 1. Because of the large mass of data, all the other Computer Outputs are abbreviated with only the pertinent data being presented.

Observing Computer Output Number 1 for the reduced Bona system, we see first that it is unobservable because of the zero value in the observability functions. Observing the eigenvector for the zero eigenvalue, we also see that state-variables numbered three, five, and six are unobservable because a component of the eigenvector is in the direction of each of these state-variables.

For state-variable number 1, we find the value of both standard-deviation observability and upper-bound observability to be unity. This value indicates that the error of the calculated state is the same as the observation error. The reason for this result can be found by examining the C matrix in Table 1. State-variable number 1 is measured directly.

Table 1. The A and C matrix from system due to Bona (5)

$$\begin{bmatrix}
0 & \Omega_z & 0 & \alpha & 0 & 0 & \alpha & 0 & 0 \\
-\Omega_z & 0 & \Omega_x & 0 & \alpha & 0 & 0 & \alpha & 0 \\
0 & -\Omega_x & 0 & 0 & 0 & \alpha & 0 & 0 & \alpha \\
0 & 0 & 0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\beta_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

A matrix (9 x 9)

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

C matrix (2 x 9)

Table 2. The A and C matrix from system due to Brown (7)

$$
\begin{bmatrix}
0 & \Omega_z & 0 & 0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Omega_z & 0 & \Omega_z & 0 & 0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\Omega_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\omega_o & 0 & 0 & -\omega_o & 0 & 0 & 2\Omega_z & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega_o & 0 & 0 & -2\Omega_z & -\omega_o & 0 & 0 & 0 & 0 & 0 & \omega_o & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta_9
\end{bmatrix}
$$

A matrix (16 x 16)

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
C_{\mu x} & C_{\mu y} & C_{\mu z} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
C_{vx} & C_{vy} & C_{vz} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

C matrix (6 x 16)

Table 3. Values used in the calculations

$$\Omega_x = 0.5156 \times 10^{-4}$$

$$\Omega_z = 0.5156 \times 10^{-4}$$

$$\alpha = 1.0 \times 10^{-4}$$

$$\omega_o = 0.124 \times 10^{-2}$$

$$\beta_1 = 0.278 \times 10^{-4}$$

$$\beta_2 = 0.278 \times 10^{-4}$$

$$\beta_3 = 0.278 \text{ z } 10^{-4}$$

$$\beta_4 = 0.278 \times 10^{-4}$$

$$\beta_5 = 0.278 \times 10^{-4}$$

$$\beta_6 = 0.278 \times 10^{-3}$$

$$\beta_7 = 0.278 \times 10^{-3}$$

$$\beta_8 = 0.556 \times 10^{-3}$$

$$\beta_9 = 0.556 \times 10^{-3}$$

$C_{\mu x}$, $C_{\mu y}$, $C_{\mu z}$, C_{vx}, C_{vy}, C_{vz} are functions of time and are defined in Appendix C.

```
OBSERVABILITY FUNCTIONS
 0.0             0.1517D-01      0.2210D-01      0.4589D 00
 0.5537D 01      0.5967D 01
STATE VARIABLE NO. ON LEFT MARGIN
     SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
     0.0        0.01517    0.02210    0.45891    5.53654    5.96728
 1   0.0        0.00065    0.00139    0.64027    0.76815    0.00140
 2   0.0        0.08927   -0.06539    0.00024   -0.00196    0.99386
 3  -0.16830    0.04227    0.06171    0.75494   -0.62941   -0.00116
 4   0.0       -0.27201    0.95709   -0.03791    0.02993    0.08748
 5   0.86782   -0.44761   -0.12128    0.13506   -0.11205    0.03197
 6   0.46751    0.84611    0.24734    0.02104   -0.01859   -0.05977
DEGREE OF OBSERVABILITY PER STATE VARIABLE
   STANDARD   UPPER    OBSERVED OUTPUT NO. AND PROPORTIONAL
   DEVIATION  BOUND    PART OF THAT VALUE IF PART IS OVER 0.1
 1  1.0000    1.0000   ( 1, 1.00)
 2  1.0634    0.5989   ( 2, 0.53)( 4,-0.16)
 3 NOT OBSERVABLE
 4  0.1469    0.0700   ( 2,-0.31)( 3, 0.31)( 4, 0.12)( 5,-0.12)
 5 NOT OBSERVABLE
 6 NOT OBSERVABLE
```

Computer Output No. 1. Bona's system with state variables no. 7, 8, 9 omitted

```
OBSERVABILITY FUNCTIONS
 0.0              0.0              0.1127D-02       0.1476D-02
 0.2979D-01       0.3755D-01       0.4861D 00       0.8509D 01
 0.8935D 01
STATE VARIABLE NO. ON LEFT MARGIN
     SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
     0.0        0.0        0.00113    0.00148    0.02979    0.03755
 1   0.0        0.0        0.00003   -0.00005    0.00098    0.00160
 2   0.0        0.0       -0.00496   -0.00417    0.13043   -0.10372
 3  -0.16806   -0.18532    0.02677   -0.03249    0.03180    0.04111
 4   0.0        0.0       -0.12075   -0.66380   -0.09471    0.72549
 5   0.86786   -0.02667   -0.31139    0.02638   -0.33796   -0.06518
 6   0.46753   -0.01437    0.58766   -0.06066    0.63880    0.13580
 7   0.0        0.0        0.12705    0.72325   -0.17597    0.64864
 8  -0.00131    0.98221    0.00519   -0.00630    0.00617    0.00797
 9   0.0        0.0       -0.72540    0.17548    0.64785    0.13312
DEGREE OF OBSERVABILITY PER STATE VARIABLE
   STANDARD   UPPER    OBSERVED OUTPUT NO. AND PROPORTIONAL
   DEVIATION  BOUND    PART OF THAT VALUE IF PART IS OVER 0.1
 1  1.0000    1.0000   ( 1, 1.00)
 2  1.0000    1.0000   ( 2, 1.00)
 3 NOT OBSERVABLE
 4  0.0554    0.0262   ( 4, 0.31)( 5,-0.31)( 6,-0.12)( 7, 0.12)
 5 NOT OBSERVABLE
 6 NOT OBSERVABLE
 7  0.0512    0.0206   ( 2,-0.11)( 3, 0.11)( 4,-0.24)( 5, 0.24)
 8 NOT OBSERVABLE
 9  0.0446    0.0126   ( 4, 0.15)( 5, 0.15)
```

Computer Output No. 2. Bona's full system

For state-variable number 2, we find the value for the standard-deviation observability to be greater than unity. An examination of the C matrix in Table 1 reveals that this state-variable is also measured directly but the error in the calculated state variable is less than the measurement error. The calculated error is less because information from more than one observation is used in the calculation of the state variable with the result that the upper-bound error observability is much less than unity. We must remember that the generalized inverse was used to obtain these degrees of observabilities and that it optimizes in the least square sense. In other words, it gives us the largest value for the standard-deviation observability, but not for the upper-bound observability.

State-variable number 4 has a standard-deviation error 6.8 times the standard-deviation error in the measurement.

Examining the results of the full Bona system reveals that state-variable number 2 is determined only by the direct measurement on it instead of a number of measurements as was the case in the reduced Bona system.

The sixteen state-variable system due to Brown was run with various combinations of the output terminals being observed. The combinations of outputs being observed are listed at the top of each Computer Output Number 3 through 13.

When outputs number 3 and 4 were observed, special difficulty was encountered in the formation of the Q matrix because the C matrix contains time varying functions. The derivative of each function had to be taken 15 times. The derivatives were formed on the computer by algebraic means rather than by numerical techniques. The details of how this was done is given in Appendix C.

A better feeling for the various criteria for the degrees of observability can be obtained by studying the Computer Outputs Number 3 through 13. Since the Computer Outputs are fairly straightforward, no further discussion will be given here except to explain how the number of the output is specified.

In the lower right hand part of the Computer Output, the first number inside the parenthesis is the number of the system output observed. The numbers 1 through 6 are the direct observations, the numbers 7 through 12 are the observations of the first derivative of the system outputs 1 through 6 respectively, and so forth, for the rest of the output numbers.

```
OBSERVED OUTPUT NOS.1,2
OBSERVABILITY FUNCTIONS
 0.0                 0.0                 0.0                 0.0
 0.0                 0.0                 0.1077D-07          0.3128D-04
 0.4248D-02          0.4720D-02          0.7198D 00          0.7205D 00
 0.3432D 01          0.3433D 01          0.1184D 02          0.1184D 02
STATE VARIABLE NO. ON LEFT MARGIN
       SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
       0.0          0.0          0.0          0.0          0.0          0.0
  1    0.0          0.0         -0.00017      0.00030      0.41260     -0.70396
  2    0.0          0.0          0.00026      0.70641     -0.00020      0.00018
  3    0.0          0.0          0.99889     -0.00037      0.00093     -0.00157
  4    0.0          0.0         -0.00017     -0.00007      0.40359      0.70944
  5    0.0          0.0          0.0         -0.00000      0.0          0.0
  6    0.0          0.0          0.0          0.00000      0.0          0.0
  7    0.0          0.0          0.0          0.0          0.0          0.0
  8    0.0          0.0         -0.00000     -0.02938     -0.01829     -0.00013
  9    0.0          0.0         -0.04155     -0.01581      0.01712     -0.02921
 10    0.0          0.0         -0.02238      0.02938     -0.00908     -0.01586
 11    0.0          0.0         -0.00035      0.00023      0.81619      0.00549
 12    0.0          0.0          0.00026      0.70641     -0.00020      0.00018
 13    0.0          0.0          0.0          0.00000      0.0          0.0
 14    0.0          0.0          0.0          0.0          0.0          0.0
 15    0.0          1.00000      0.0          0.0          0.0          0.0
 16    1.00000      0.0          0.0          0.0          0.0          0.0
DEGREE OF OBSERVABILITY PER STATE VARIABLE
    STANDARD    UPPER    OBSERVED OUTPUT NO. AND PROPORTIONAL
    DEVIATION   BOUND    PART OF THAT VALUE IF PART IS OVER 0.05
  1 NOT OBSERVABLE
  2 NOT OBSERVABLE
  3 NOT OBSERVABLE
  4 NOT OBSERVABLE
  5 NOT OBSERVABLE
  6 NOT OBSERVABLE
  7  0.0290      0.0090   (13,-0.10)(14,-0.25)(20, 0.07)(26,-0.08)
                          (38, 0.05)
  8 NOT OBSERVABLE
  9 NOT OBSERVABLE
 10 NOT OBSERVABLE
 11 NOT OBSERVABLE
 12 NOT OBSERVABLE
 13 NOT OBSERVABLE
 14  0.0290      0.0091   (13, 0.10)(14, 0.25)(20,-0.07)(26, 0.08)
                          (38,-0.05)
 15 NOT OBSERVABLE
 16 NOT OBSERVABLE
```

Computer Output No. 3. Brown's system observing output no. 1 and 2

```
OBSERVED OUTPUT NOS.5,6
OBSERVABILITY FUNCTIONS
 0.C                0.0                0.0                C.0
 0.0                C.0                0.0                0.4772D-C3
 C.4443D CC         C.4454D C0         0.6253D C0         0.6258D C0
 0.3691D 01         C.3692D 01         0.1124D 02         0.1124D 02
STATE VARIABLE NO. ON LEFT MARGIN
       SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
       C.0          C.0          C.0          C.0          0.C          C.0
 1     C.0          C.C          C.0          0.0         -0.C0028      0.CC019
 2     C.0          C.C          0.0          C.0          0.C0047      C.70641
 3     C.0          0.C          C.0          C.0          0.99889     -0.CCC66
 4     C.0          0.0          0.0          0.0          0.0          C.0
 5     C.0          0.0          0.0          C.0          C.0          0.0
 6     C.C          C.C          C.0          0.0          0.0          0.0
 7     C.0          0.C          C.0          0.0          0.0          0.0
 8     C.0          0.0          0.0          C.0         -0.00001     -C.02938
 9     C.0          0.0          C.0          C.0         -C.04156     -0.01580
10     C.0          0.0          0.0          0.0         -C.C2237      0.02939
11     C.0          0.C          C.0          0.0         -0.0002      0.CC019
12     C.0          0.C          C.0          C.0          0.C0047      C.70641
13     C.0          0.0          0.0          1.0000C      0.C          0.0
14     C.0          0.0          1.00000      0.0          0.0          0.0
15     C.0          1.CCC00      C.0          0.0          0.0          C.0
16     1.CCOC0      C.0          0.0          C.0          0.0          C.0
DEGREE OF OBSERVABILITY PER STATE VARIABLE
     STANDARD     UPPER     OBSERVED OUTPUT NO. AND PROPORTIONAL
     DEVIATION    BOUND     PART OF THAT VALUE IF PART IS OVER 0.05
 1 NOT OBSERVABLE
 2 NOT OBSERVABLE
 3 NOT OBSERVABLE
 4   1.CCCO       1.CCCC    ( 5, 1.C0)
 5   1.0001       C.9743    (11, 0.97)
 6   1.C000       C.9989    ( 6, 1.00)
 7   1.0002       0.9542    (12, 0.95)
 8 NOT OBSERVABLE
 9 NOT OBSERVABLE
10 NOT OBSERVABLE
11 NOT OBSERVABLE
12 NOT OBSERVABLE
13 NOT OBSERVABLE
14 NOT OBSERVABLE
15 NOT OBSERVABLE
16 NOT OBSERVABLE
```

Computer Output No. 4. Brown's system observing outputs no. 5 and 6

```
OBSERVED OUTPUT NOS.1,2,5,6
OBSERVABILITY FUNCTIONS
 0.0                0.0                0.0                0.0
 0.0                0.9388D-03         0.2567D 00         0.2570D 00
 0.6443D 00         0.6448D 00         0.9391D 00         0.9406D 00
 0.7089D 01         0.7091D 01         0.2307D 02         0.2307D 02
STATE VARIABLE NO. ON LEFT MARGIN
      SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
        0.0         0.0         0.0          0.0          0.0         0.00094
  1  -0.00004     0.00018     0.0          0.0          0.70671     0.00016
  2   0.70641     0.00046     0.0          0.0          0.00004    -0.00334
  3  -0.00066     0.99889     0.0          0.0          0.00096     0.02228
  4   0.0         0.0         0.0          0.0          0.0        -0.00002
  5   0.0         0.0         0.0          0.0          0.0        -0.00103
  6   0.0         0.0         0.0          0.0          0.0        -0.00004
  7   0.0         0.0         0.0          0.0          0.0         0.00197
  8  -0.02937    -0.00002     0.0          0.0         -0.01585     0.00245
  9  -0.01581    -0.04154     0.0          0.0          0.02935    -0.00271
 10   0.02939    -0.02238     0.0          0.0         -0.00002     0.99898
 11  -0.00004     0.00018     0.0          0.0          0.70671     0.00001
 12   0.70641     0.00046     0.0          0.0          0.00004    -0.03816
 13   0.0         0.0         0.0          0.0          0.0         0.00119
 14   0.0         0.0         0.0          0.0          0.0        -0.00691
 15   0.0         0.0         0.0          1.00000      0.0         0.0
 16   0.0         0.0         1.00000      0.0          0.0         0.0
DEGREE OF OBSERVABILITY PER STATE VARIABLE
    STANDARD    UPPER     OBSERVED OUTPUT NO. AND PROPORTIONAL
    DEVIATION   BOUND     PART OF THAT VALUE IF PART IS OVER 0.05
  1 NOT OBSERVABLE
  2 NOT OBSERVABLE
  3 NOT OBSERVABLE
  4   1.0000      1.0000   ( 5, 1.00)
  5   1.0067      0.7567   ( 7, 0.08)(11, 0.75)
  6   1.0000      0.9977   ( 6, 1.00)
  7   1.0052      0.7737   (12, 0.77)
  8 NOT OBSERVABLE
  9 NOT OBSERVABLE
 10 NOT OBSERVABLE
 11 NOT OBSERVABLE
 12 NOT OBSERVABLE
 13   0.5905      0.3018   ( 1, 0.41)( 7,-0.10)(11,-0.29)
 14   0.5856      0.3279   ( 2, 0.45)( 8,-0.06)(12,-0.32)(18, 0.06)
 15 NOT OBSERVABLE
 16 NOT OBSERVABLE
```

Computer Output No. 5. Brown's system observing outputs no. 1, 2, 5 and 6

```
OBSERVED OUTPUT NOS.1,2,3,4        TIME = 6:10 A.M.
OBSERVABILITY FUNCTIONS
 0.2900D-05      0.2926D-05      0.1862D 00      0.2644D 00
 0.4760D 00      0.8265D 00      0.9189D 00      0.9770D 00
 0.1330D 01      0.1591D 01      0.3977D 01      0.4609D 01
 0.1133D 02      0.1181D 02      0.1256D 02      0.1313D 02
STATE VARIABLE NO. ON LEFT MARGIN
      SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
```

	0.00000	0.00000	0.18616	0.26441	0.47597	0.82645
1	0.00000	-0.00004	0.00351	0.11906	-0.01361	0.03742
2	0.00000	-0.00000	-0.00562	-0.00228	-0.00890	-0.00975
3	-0.00001	0.00000	0.00319	-0.01637	0.99573	0.05838
4	0.13077	0.69434	-0.04217	-0.09592	0.00593	-0.02478
5	-0.00267	-0.01525	-0.00095	-0.42420	-0.00195	-0.02265
6	0.69432	-0.13077	-0.02872	0.05361	0.00543	0.02615
7	-0.01513	0.00267	-0.50189	-0.00057	-0.00188	0.02380
8	-0.00000	-0.00013	0.01195	0.28835	-0.00266	0.02372
9	-0.00026	-0.00000	0.40996	-0.00904	-0.04310	-0.03640
10	-0.00004	0.00001	-0.01170	0.00731	-0.05769	0.99180
11	0.13207	0.69490	0.04161	0.06864	-0.00647	0.02507
12	0.69492	-0.13207	0.00183	-0.05293	-0.00582	-0.02665
13	0.00260	0.01485	0.01009	0.83802	0.02267	-0.04034
14	0.01473	-0.00260	0.75832	-0.01908	0.01719	0.04917
15	-0.00001	-0.00000	0.01271	0.00629	0.00039	-0.00574
16	0.00000	-0.00000	-0.00705	0.00801	-0.04585	0.04174

Computer Output No. 6. Brown's system observing outputs no. 1, 2, 3, 4
Time = 6:10 AM

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	1.4103	0.5380	(3, 0.19)(9, 0.11)(82, 0.23)(88,-0.20)
2	1.5574	0.7600	(75,-0.08)(81,-0.30)(87, 0.23)(93, 0.29)
3	0.6916	0.3941	(4, 0.56)
4	0.0024	0.0007	(1,-0.05)(7,-0.26)(13, 0.07)(19,-0.06)
5	0.1097	0.0378	(1, 0.06)(7, 0.30)(13,-0.09)(19, 0.07)
6	0.0024	0.0007	(2,-0.05)(8,-0.26)(14, 0.06)(20,-0.06)
7	0.1097	0.0377	(2, 0.06)(8, 0.30)(14,-0.09)(20, 0.07)
8	1.2014	0.4909	(3, 0.09)(9, 0.31)(82, 0.11)(94,-0.22)
9	0.8546	0.3290	(15,-0.08)(16, 0.08)(81,-0.10)(87,-0.32)
			(93, 0.12)
10	0.9093	0.5338	(10, 0.58)
11	0.0024	0.0007	(1,-0.05)(7,-0.26)(13, 0.07)(19,-0.06)
12	0.0024	0.0007	(2,-0.05)(8,-0.26)(14, 0.06)(20,-0.06)
13	0.1113	0.0400	(7,-0.31)(13, 0.09)(19,-0.07)(31, 0.05)
14	0.1113	0.0398	(8,-0.31)(14, 0.09)(20,-0.07)(32, 0.05)
15	3.3411	0.8524	(15, 0.07)(21,-0.08)(27, 0.08)(33,-0.08)
			(39, 0.08)(45,-0.08)(51, 0.08)(57,-0.08)
			(63, 0.08)(69,-0.08)(75, 0.07)
16	3.3105	0.8972	(16, 0.07)(22,-0.08)(28, 0.08)(34,-0.08)
			(40, 0.08)(46,-0.08)(52, 0.08)(58,-0.08)
			(64, 0.08)(70,-0.08)(76, 0.08)

Computer Output No. 6 (Continued)

```
OBSERVED OUTPUT NOS.1,2,3,4        TIME = 8:00 A.M.
OBSERVABILITY FUNCTIONS
 0.2006D-07      0.2615D-05      0.2568D-04      0.2979D-01
 0.2073D 00      0.3124D 00      0.4595D 00      0.6341D 00
 0.8992D 00      0.9624D 00      0.3561D 01      0.3691D 01
 0.1193D 02      0.1201D 02      0.1462D 02      0.1467D 02
STATE VARIABLE NO. ON LEFT MARGIN
     SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
```

	0.00000	0.00000	0.00003	0.02979	0.20735	0.31241
1	-0.09862	-0.00213	0.27752	-0.01489	-0.00423	-0.73339
2	-0.24156	-0.00534	0.67983	-0.00220	-0.00497	0.29900
3	0.09862	0.00214	-0.27754	0.01191	0.01497	-0.00180
4	0.53213	0.48656	0.03560	-0.01476	-0.01525	0.39912
5	0.00028	-0.00970	-0.03488	-0.17240	-0.46362	-0.00975
6	-0.33805	0.51607	-0.49858	-0.02254	0.05387	-0.16213
7	0.00474	-0.00985	0.01694	-0.53267	0.14347	0.00551
8	0.00000	-0.00066	-0.00023	0.17958	0.36169	0.02000
9	0.00001	-0.00177	0.00127	0.52202	-0.12395	-0.00627
10	-0.00000	0.00066	-0.00031	-0.21273	0.00180	0.01904
11	0.43302	0.48575	0.31394	0.00931	-0.01051	-0.39864
12	-0.58070	0.51044	0.18015	0.00300	-0.04528	0.16277
13	-0.00028	0.00945	0.03400	0.17597	0.74430	0.00328
14	-0.00474	0.00959	-0.01613	0.55158	-0.24202	-0.00108
15	0.00000	-0.00003	-0.00000	0.00802	0.01251	0.02846
16	-0.00000	0.00003	-0.00001	-0.00865	-0.00237	0.00079

Computer Output No. 7. Brown's system observing outputs no. 1, 2, 3, 4
Time = 8:00 AM

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	0.0014	0.0004	(8, 0.07)(13, 0.08)(14, 0.23)(26, 0.07)
2	0.0006	0.0002	(8, 0.07)(13, 0.08)(14, 0.23)(26, 0.07)
3	0.0014	0.0004	(8,-0.07)(13,-0.08)(14,-0.23)(26,-0.07)
4	0.0003	0.0001	(8,-0.07)(13,-0.07)(14,-0.22)(26,-0.07)
5	0.1056	0.0324	(1, 0.05)(7, 0.25)(13,-0.09)(14,-0.09)
			(19, 0.06)
6	0.0004	0.0001	(13, 0.07)(14, 0.23)(20,-0.06)(26, 0.07)
			(38,-0.05)
7	0.0291	0.0093	(13,-0.08)(14,-0.27)(20, 0.07)(26,-0.08)
			(38, 0.06)(50,-0.05)
8	0.6575	0.2089	(9, 0.16)(15,-0.19)(16, 0.18)
9	0.3031	0.0985	(9,-0.10)(15,-0.23)(16, 0.19)
10	0.5807	0.1848	(10, 0.20)(15, 0.15)(16,-0.17)
11	0.0003	0.0001	(7,-0.05)(8,-0.07)(13,-0.07)(14,-0.22)
			(26,-0.07)
12	0.0002	0.0001	(13, 0.07)(14, 0.23)(20,-0.05)(26, 0.07)
13	0.1071	0.0339	(7,-0.26)(13, 0.10)(14, 0.09)(19,-0.06)
14	0.0292	0.0095	(13, 0.08)(14, 0.27)(20,-0.07)(26, 0.08)
			(38,-0.06)(50, 0.05)
15	3.6563	0.9196	(21,-0.07)(27, 0.07)(33,-0.07)(39, 0.07)
			(45,-0.07)(51, 0.07)(57,-0.07)(63, 0.07)
			(69,-0.07)(75, 0.07)(81,-0.07)(87, 0.07)
			(93,-0.07)
16	3.6422	0.9185	(22,-0.06)(28, 0.07)(34,-0.07)(40, 0.07)
			(46,-0.07)(52, 0.07)(58,-0.07)(64, 0.07)
			(70,-0.07)(76, 0.07)(82,-0.07)(88, 0.07)
			(94,-0.07)

Computer Output No. 7 (Continued)

```
OBSERVED OUTPUT NOS.1,2,3,4        TIME =10:00 A.M.
OBSERVABILITY FUNCTIONS
```

0.1122D-07	0.2712D-05	0.3731D-04	0.3472D-01
0.2291D 00	0.3117D 00	0.4161D 00	0.4922D 00
0.1006D 01	0.1016D 01	0.3684D 01	0.3777D 01
0.1198D 02	0.1205D 02	0.1449D 02	0.1451D 02

```
STATE VARIABLE NO. ON LEFT MARGIN
     SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
```

	0.00000	0.00000	0.00004	0.03472	0.22914	0.31167
1	0.14888	-0.00082	-0.51252	-0.00754	-0.00188	-0.50131
2	0.12156	-0.00073	-0.41848	0.00941	0.00071	0.61274
3	-0.14888	0.00082	0.51258	0.00963	-0.04712	-0.00140
4	-0.29675	0.66791	0.19090	-0.02341	-0.00914	0.27128
5	-0.00169	-0.01366	-0.02482	-0.36312	0.33286	-0.01509
6	0.58065	0.23213	0.39380	-0.01952	-0.05661	-0.33125
7	-0.00333	-0.00449	-0.04037	-0.35877	-0.32597	0.00358
8	-0.00000	-0.00100	0.00012	0.36305	-0.24232	0.01979
9	-0.00000	-0.00089	-0.00077	0.34550	0.25433	-0.01191
10	0.00001	0.00111	0.00015	-0.44912	0.00088	0.06551
11	-0.14719	0.66818	-0.32104	0.01033	0.02821	-0.27125
12	0.70284	0.23053	-0.02271	0.00515	0.03701	0.33119
13	0.00169	0.01330	-0.02434	0.37736	-0.56691	0.02501
14	0.00333	0.00437	0.03901	0.37770	0.57479	0.01051
15	-0.00000	-0.00002	0.00001	0.00740	-0.00795	0.02911
16	0.00000	0.00003	-0.00001	-0.01142	0.00607	0.00164

Computer Output No. 8. Brown's system observing outputs no. 1, 2, 3, 4
Time = 10:00 AM

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	0.0007	0.0002	(8, 0.05)(13, 0.16)(14, 0.15)(26, 0.05)
2	0.0009	0.0002	(8, 0.05)(13, 0.16)(14, 0.15)(26, 0.05)
3	0.0007	0.0002	(8,-0.05)(13,-0.16)(14,-0.15)(26,-0.05)
4	0.0004	0.0001	(7,-0.08)(13,-0.14)(14,-0.14)(26,-0.05)
5	0.0539	0.0149	(7, 0.07)(13,-0.18)(14,-0.15)
6	0.0002	0.0000	(13, 0.16)(14, 0.16)(20,-0.05)(26, 0.06)
7	0.0309	0.0086	(13,-0.15)(14,-0.18)(20, 0.07)(26,-0.06)
8	0.4558	0.1301	(15,-0.22)(16, 0.12)(22,-0.06)(93,-0.06)
9	0.4752	0.1368	(9,-0.14)(15,-0.20)(16, 0.05)(22,-0.08) (94,-0.06)
10	0.3626	0.1000	(9, 0.05)(10, 0.15)(15, 0.16)(16,-0.10) (22, 0.06)
11	0.0007	0.0002	(7,-0.10)(13,-0.12)(14,-0.13)(26,-0.05)
12	0.0002	0.0000	(13, 0.16)(14, 0.16)(20,-0.05)(26, 0.06)
13	0.0540	0.0152	(7,-0.08)(13, 0.18)(14, 0.15)
14	0.0310	0.0087	(13, 0.15)(14, 0.18)(20,-0.07)(26, 0.06)
15	3.6681	0.9164	(21,-0.07)(27, 0.07)(33,-0.07)(39, 0.07) (45,-0.07)(51, 0.07)(57,-0.07)(63, 0.07) (69,-0.07)(75, 0.07)(81,-0.07)(87, 0.06) (93,-0.07)
16	3.6003	0.8933	(22,-0.05)(28, 0.07)(34,-0.07)(40, 0.07) (46,-0.07)(52, 0.07)(58,-0.07)(64, 0.07) (70,-0.07)(76, 0.07)(82,-0.07)(88, 0.07) (94,-0.06)

Computer Output No. 8 (Continued)

OBSERVED OUTPUT NOS.1,2,3,4 TIME =12:10 P.M.

OBSERVABILITY FUNCTIONS

0.7771D-08	0.2871D-05	0.4283D-04	0.7806D-01
0.3061D 00	0.3230D 00	0.3944D 00	0.4594D 00
0.1411D 01	0.1467D 01	0.4265D 01	0.4362D 01
0.1204D 02	0.1220D 02	0.1333D 02	0.1337D 02

STATE VARIABLE NO. ON LEFT MARGIN

SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS

	0.00000	0.00000	0.00004	0.07806	0.30614	0.32301
1	0.16439	0.00182	-0.60857	0.00004	-0.06777	-0.04002
2	0.01015	0.00007	-0.03756	-0.02077	0.71667	-0.32626
3	-0.16439	-0.00183	0.60870	-0.02106	-0.05166	-0.13073
4	-0.07044	0.70552	0.30911	0.02233	0.01176	-0.02972
5	-0.00249	-0.01523	0.00834	0.40881	-0.01538	0.04182
6	0.67742	-0.01264	0.20104	0.00763	-0.39845	0.14119
7	-0.00173	0.00023	-0.05320	0.02448	-0.14502	-0.32433
8	-0.00000	-0.00042	0.00067	-0.38548	0.01575	-0.01473
9	-0.00000	-0.00002	-0.00028	-0.01386	0.07336	0.18485
10	0.00000	0.00064	-0.00125	0.67967	0.10380	-0.05008
11	0.09470	0.70812	-0.29763	-0.00456	-0.01261	0.03167
12	0.68775	-0.01377	0.16593	-0.00603	0.38717	-0.16579
13	0.00249	0.01483	-0.00839	-0.46762	0.04943	-0.07906
14	0.00173	-0.00023	0.05165	-0.03204	0.35932	0.82167
15	-0.00000	-0.00001	0.00001	-0.00879	0.02970	-0.01327
16	0.00000	0.00002	-0.00005	0.01694	0.00738	0.01225

Computer Output No. 9. Brown's system observing outputs no. 1, 2, 3, 4
Time = 12:10 PM

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	0.0005	0.0002	(7, 0.08)(13, 0.25)(25, 0.07)
2	0.0087	0.0026	(7, 0.08)(13, 0.25)(25, 0.06)
3	0.0005	0.0002	(7,-0.08)(13,-0.25)(25,-0.07)
4	0.0011	0.0003	(7,-0.17)(13,-0.18)
5	0.0337	0.0113	(13,-0.29)(14,-0.06)(19, 0.05)(20, 0.05) (25,-0.08)(37, 0.05)(49,-0.05)
6	0.0001	0.0000	(7, 0.08)(13, 0.25)(14, 0.05)(20,-0.05) (25, 0.07)
7	0.0470	0.0118	(7,-0.06)(8, 0.08)(13,-0.18)(14,-0.06) (20, 0.06)(25,-0.06)
8	0.6688	0.1760	(10,-0.13)(15,-0.09)(82, 0.08)(87,-0.08) (93,-0.07)(94,-0.13)
9	1.5235	0.4372	(9,-0.17)(16,-0.08)(81,-0.06)(87,-0.05) (88,-0.09)(93, 0.13)(94,-0.10)
10	0.3791	0.1255	(10, 0.28)(15, 0.07)(22, 0.05)(82,-0.07) (87, 0.05)(94, 0.10)
11	0.0009	0.0003	(13, 0.26)(14, 0.05)(19,-0.05)(20,-0.05) (25, 0.07)(37,-0.05)(49, 0.05)
12	0.0001	0.0000	(7, 0.08)(13, 0.25)(14, 0.05)(20,-0.05) (25, 0.07)
13	0.0337	0.0114	(13, 0.29)(14, 0.06)(19,-0.05)(20,-0.06) (25, 0.08)(37,-0.06)(49, 0.05)
14	0.0471	0.0120	(7, 0.06)(8,-0.08)(13, 0.18)(14, 0.06) (20,-0.06)(25, 0.06)
15	3.5042	0.8722	(21,-0.07)(27, 0.07)(33,-0.07)(39, 0.07) (45,-0.07)(51, 0.07)(57,-0.07)(63, 0.07) (69,-0.07)(75, 0.07)(81,-0.06)
16	3.4429	0.8576	(22,-0.06)(28, 0.07)(34,-0.07)(40, 0.07) (46,-0.07)(52, 0.07)(58,-0.07)(64, 0.07) (70,-0.07)(76, 0.08)(82,-0.08)

Computer Output No. 9 (Continued)

OBSERVED OUTPUT NOS.1,2,3,4,5,6 TIME = 6:10 A.M.

OBSERVABILITY FUNCTIONS

0.3468D 00	0.3592D 00	0.4072D 00	0.4356D 00
0.4763D 00	0.8286D 00	0.1254D 01	0.1330D 01
0.1783D 01	0.2106D 01	0.7527D 01	0.7992D 01
0.1154D 02	0.1192D 02	0.2363D 02	0.2407D 02

STATE VARIABLE NO. ON LEFT MARGIN

SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS

	0.34682	0.35924	0.40717	0.43565	0.47628	0.82856
1	0.06922	0.15323	0.05939	0.08495	-0.00270	0.01431
2	0.00864	-0.02328	0.15421	-0.05261	-0.00192	0.00649
3	-0.02117	-0.02512	-0.04053	-0.01118	0.99460	0.05912
4	0.14000	0.27062	0.17285	0.41851	0.02961	-0.01865
5	-0.06534	-0.20581	0.02526	0.11117	-0.00044	-0.00472
6	0.04144	-0.09033	0.51846	-0.18456	0.02072	0.03194
7	-0.28415	0.09564	0.04376	0.00462	-0.00101	0.00839
8	0.06473	0.16701	0.00628	-0.09630	-0.00001	0.00814
9	0.25600	-0.10420	-0.06451	-0.01859	-0.04463	-0.01404
10	-0.00906	0.01752	-0.01555	0.01349	-0.05812	0.99614
11	0.26197	0.47583	0.25394	0.55161	0.03228	-0.00365
12	0.07153	-0.15886	0.75553	-0.23914	0.02237	0.00616
13	0.20340	0.67199	-0.09024	-0.62853	0.01413	-0.00839
14	0.83867	-0.31630	-0.14802	-0.04339	0.01655	0.02139
15	0.00767	-0.00376	-0.00095	-0.00978	0.00011	-0.00425
16	-0.00054	0.01074	0.00432	0.00382	-0.04535	0.04045

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	1.4540	0.4504	(3, 0.15)(9, 0.06)(11,-0.06)(82, 0.18)(88,-0.17)(94, 0.07)
2	1.5578	0.7402	(75,-0.08)(81,-0.29)(87, 0.22)(93, 0.29)
3	0.6917	0.3861	(4, 0.55)
4	1.0001	0.9406	(5, 0.94)
5	1.4520	0.3565	(9,-0.12)(11, 0.17)(94, 0.08)
6	1.0001	0.9477	(6, 0.95)
7	1.2838	0.3615	(12, 0.22)(87, 0.14)(93,-0.05)
8	1.5105	0.3960	(9, 0.16)(11,-0.14)(82, 0.06)(94,-0.11)
9	1.2460	0.3082	(12,-0.16)(87,-0.14)(93, 0.05)
10	0.9099	0.5038	(10, 0.54)
11	0.7578	0.1769	(3, 0.06)(5, 0.18)(82, 0.07)(88,-0.07)
12	0.7717	0.2068	(6, 0.21)(81,-0.08)(87, 0.06)(93, 0.06)
13	0.6490	0.2185	(1, 0.30)(7,-0.06)(9, 0.07)(11,-0.10)
14	0.6322	0.2205	(2, 0.30)(8,-0.06)(12,-0.13)(87,-0.08)
15	3.3572	0.8190	(15, 0.07)(21,-0.07)(27, 0.07)(33,-0.07)(39, 0.07)(45,-0.07)(51, 0.07)(57,-0.07)(63, 0.07)(69,-0.07)(75, 0.06)
16	3.3169	0.8716	(16, 0.07)(22,-0.08)(28, 0.08)(34,-0.08)(40, 0.08)(46,-0.08)(52, 0.08)(58,-0.08)(64, 0.08)(70,-0.08)(76, 0.08)

Computer Output No. 10. Brown's system observing all outputs
Time = 6:10 AM

```
OBSERVED OUTPUT NOS.1,2,3,4,5,6  TIME = 8:00 A.M.
OBSERVABILITY FUNCTIONS
 0.4549D-03      0.1985D 00      0.2615D 00      0.3596D 00
 0.4283D 00      0.6752D 00      0.6812D 00      0.7796D 00
 0.1103D 01      0.1317D 01      0.7201D 01      0.7310D 01
 0.1459D 02      0.1462D 02      0.2321D 02      0.2326D 02
STATE VARIABLE NO. ON LEFT MARGIN
     SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
      0.00045   0.19850   0.26149   0.35962   0.42835   0.67523
  1   0.25775   0.51922  -0.01157  -0.02284   0.01492  -0.15001
  2   0.63106  -0.21188  -0.01143   0.02490   0.03887  -0.02877
  3  -0.25809   0.00095   0.01085  -0.00398   0.92040  -0.04690
  4  -0.00018   0.18426  -0.00177  -0.00474   0.08641   0.18875
  5  -0.01300  -0.00926  -0.12063  -0.27616   0.00074   0.04804
  6   0.00038  -0.07500   0.00548  -0.01791   0.21263   0.08862
  7   0.00405  -0.00221  -0.37591   0.08440   0.00464   0.12060
  8  -0.01617  -0.00645   0.13571   0.24898   0.01018  -0.06224
  9   0.00220   0.01121   0.39581  -0.08369  -0.05615  -0.19899
 10   0.00123  -0.01358  -0.24000   0.00341  -0.02206   0.82350
 11   0.25841   0.74321  -0.00210  -0.01629   0.11548   0.08738
 12   0.63349  -0.30371   0.02333  -0.02935   0.28324   0.03233
 13   0.01285   0.02270   0.23077   0.87789   0.01058   0.12780
 14  -0.00352  -0.00126   0.74640  -0.27236  -0.00282   0.40794
 15  -0.00056  -0.01970   0.00586   0.00973   0.00034   0.00356
 16   0.00015  -0.00059  -0.00937  -0.00133  -0.03173   0.03281
DEGREE OF OBSERVABILITY PER STATE VARIABLE
    STANDARD   UPPER    OBSERVED OUTPUT NO. AND PROPORTIONAL
    DEVIATION  BOUND    PART OF THAT VALUE IF PART IS OVER 0.05
  1  0.0822    0.0195   ( 9,-0.16)(11,-0.14)
  2  0.0338    0.0082   ( 9,-0.16)(11,-0.15)
  3  0.0821    0.0192   ( 9, 0.15)(11, 0.14)
  4  1.0000    0.9902   ( 5, 0.99)
  5  1.0105    0.5978   ( 7, 0.06)(11, 0.59)
  6  1.0000    0.9754   ( 6, 0.98)
  7  1.0454    0.4691   (12, 0.43)(15, 0.08)(16,-0.07)
  8  0.9053    0.2832   ( 9, 0.28)(12,-0.08)(15,-0.06)(16, 0.07)
  9  0.9717    0.2287   ( 9,-0.07)(12,-0.19)
 10  0.8183    0.2862   (10, 0.32)(12, 0.09)(16,-0.07)
 11  0.0817    0.0185   ( 9,-0.15)(11,-0.14)
 12  0.0337    0.0080   ( 9,-0.16)(11,-0.14)
 13  0.5913    0.2784   ( 1, 0.38)( 7,-0.09)(11,-0.26)
 14  0.5978    0.2547   ( 2, 0.34)( 8,-0.08)(12,-0.23)
 15  3.7125    0.9227   (15, 0.06)(21,-0.07)(27, 0.07)(33,-0.07)
                        (39, 0.07)(45,-0.07)(51, 0.07)(57,-0.07)
                        (63, 0.07)(69,-0.07)(75, 0.07)(81,-0.07)
                        (87, 0.07)(93,-0.07)
 16  3.7029    0.9225   (16, 0.05)(22,-0.07)(28, 0.07)(34,-0.07)
                        (40, 0.07)(46,-0.07)(52, 0.07)(58,-0.07)
                        (64, 0.07)(70,-0.07)(76, 0.07)(82,-0.07)
                        (88, 0.07)(94,-0.07)
```

Computer Output No. 11. Brown's system observing all outputs
Time = 8:00 AM

```
OBSERVED OUTPUT NOS.1,2,3,4,5,6  TIME =10:00 A.M.
OBSERVABILITY FUNCTIONS
 0.7811D-03      0.1983D 00      0.2254D 00      0.3493D 00
 0.3659D 00      0.5142D 00      0.7217D 00      0.7796D 00
 0.1294D 01      0.1395D 01      0.7299D 01      0.7383D 01
 0.1443D 02      0.1447D 02      0.2327D 02      0.2331D 02
STATE VARIABLE NO. ON LEFT MARGIN
    SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS
      0.00078   0.19827   0.22536   0.34928   0.36988   0.51421
 1    0.47898  -0.35446   0.02214   0.13344   0.01067  -0.05760
 2    0.39111   0.43412  -0.01393   0.11130  -0.03278   0.04304
 3   -0.48040  -0.00255  -0.02706   0.82662   0.00868  -0.01751
 4    0.00009  -0.12550   0.00927   0.19432   0.01897   0.06137
 5   -0.01276   0.01980   0.25671   0.00160   0.19672   0.00063
 6    0.00062   0.15299  -0.01742   0.15864   0.01983  -0.04126
 7    0.01353   0.00786   0.25465   0.01012  -0.18955  -0.01323
 8   -0.01153  -0.01178  -0.28665  -0.00209  -0.16861  -0.05065
 9    0.01443  -0.02420  -0.27199  -0.06294   0.17163  -0.04317
10   -0.00107   0.06558   0.55620   0.01730  -0.00187   0.71940
11    0.48073  -0.50702   0.02108   0.36194   0.03908   0.04745
12    0.39332   0.61845  -0.06971   0.29471   0.02696  -0.04914
13    0.01284  -0.03587  -0.43837   0.01128  -0.66445   0.47583
14   -0.01285  -0.00508  -0.45189  -0.02187   0.64935   0.48512
15   -0.00039   0.01985  -0.00556   0.00027  -0.00677   0.00449
16    0.00026   0.00172   0.01531  -0.02731   0.00322   0.02186
```

Computer Output No. 12. Brown's system observing all outputs
Time = 10:00 AM

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	0.0582	C.0124	(9,-0.13)(11,-0.10)(12, 0.10)
2	0.C712	C.0152	(9,-0.13)(11,-0.10)(12, 0.10)
3	0.0580	C.0122	(9, 0.13)(11, 0.10)(12,-0.10)
4	1.C000	C.9709	(5, 0.97)
5	1.C398	0.4163	(11, 0.39)(15, 0.07)
6	1.C00C	C.9766	(6, 0.98)
7	1.0310	C.4727	(12, 0.44)(15, C.07)
8	1.0108	0.2087	(9, 0.11)(11,-0.10)(12,-0.09)(15,-0.06) (16, 0.07)
9	0.9736	C.2143	(9,-0.15)(11,-0.09)(12,-0.08)
10	0.6281	0.1616	(10, 0.21)(11, 0.09)(12, 0.08)(16,-0.05)
11	C.0580	C.012C	(9,-0.12)(11,-0.09)(12, C.10)
12	0.C706	C.0145	(9,-0.12)(11,-0.09)(12, 0.10)
13	0.5966	C.2384	(1, 0.32)(7,-0.08)(11,-C.21)
14	0.5954	C.2545	(2, 0.34)(8,-0.08)(12,-0.23)
15	3.7111	C.9124	(15, 0.05)(21,-0.07)(27, 0.07)(33,-0.07) (39, 0.07)(45,-0.07)(51, 0.07)(57,-0.07) (63, 0.07)(69,-0.07)(75, C.07)(81,-0.07) (87, 0.07)(93,-0.07)
16	3.6786	0.8947	(22,-0.06)(28, 0.07)(34,-0.07)(40, 0.07) (46,-0.07)(52, 0.07)(58,-0.07)(64, 0.07) (70,-0.07)(76, 0.07)(82,-0.07)(88, 0.07) (94,-0.07)

Computer Output No. 12 (Continued)

OBSERVED OUTPUT NOS.1,2,3,4,5,6 TIME =12:10 P.M.

OBSERVABILITY FUNCTIONS

0.1432D-02	0.1955D 00	0.2009D 00	0.3148D 00
0.4150D 00	0.4915D 00	0.7370D 00	0.7813D 00
0.1983D 01	0.2007D 01	0.7725D 01	0.7828D 01
0.1309D 02	0.1313D 02	0.2351D 02	0.2359D 02

STATE VARIABLE NO. ON LEFT MARGIN

SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS

	0.00143	0.19547	0.20092	0.31475	0.41504	0.49153
1	0.57352	-0.03631	0.01212	0.24141	-0.02675	-0.03704
2	0.03565	0.43012	-0.35834	0.01693	-0.03675	0.07128
3	-0.57762	-0.02232	-0.02464	0.77182	-0.01935	-0.01458
4	0.00072	-0.01215	0.00487	0.24302	0.04501	0.02359
5	-0.00481	0.18329	0.22781	0.01007	0.02034	-0.04509
6	0.00070	0.14083	-0.13861	0.01433	0.00289	-0.08613
7	0.02854	-0.00059	0.02109	0.00532	-0.19712	-0.01409
8	-0.00873	-0.20586	-0.25397	-0.02293	-0.00902	-0.04024
9	0.01208	-0.00957	-0.00605	-0.05019	0.14054	0.00254
10	0.01295	0.54286	0.54310	0.04470	-0.00251	0.50643
11	0.57692	-0.05982	0.00567	0.52841	0.06429	0.00700
12	0.03576	0.57960	-0.55140	0.03033	-0.01055	-0.08995
13	0.00539	-0.28730	-0.36431	-0.00950	-0.07677	0.84408
14	-0.02794	-0.00065	-0.04985	-0.01593	0.96240	0.06499
15	-0.00017	0.01445	-0.01751	-0.00005	-0.00144	0.00821
16	0.00096	0.01314	0.01348	-0.02829	0.00753	0.01148

DEGREE OF OBSERVABILITY PER STATE VARIABLE

	STANDARD DEVIATION	UPPER BOUND	OBSERVED OUTPUT NO. AND PROPORTIONAL PART OF THAT VALUE IF PART IS OVER 0.05
1	0.0659	0.0142	(9,-0.06)(12, 0.16)(93, 0.06)
2	0.5821	0.1503	(3,-0.21)(12, 0.10)
3	0.0652	0.0137	(9, 0.06)(12,-0.16)(93,-0.06)
4	1.0001	0.9421	(5, 0.94)
5	1.1807	0.3001	(10, 0.06)(11, 0.22)(94, 0.07)
6	1.0000	0.9808	(6, 0.98)
7	1.0208	0.5143	(8, 0.05)(12, 0.49)
8	1.0967	0.2061	(10,-0.06)(11,-0.13)(94,-0.05)
9	1.5416	0.4096	(9,-0.15)(16,-0.08)(81,-0.05)(88,-0.09) (93, 0.13)(94,-0.07)
10	0.5200	0.1392	(10, 0.21)(11, 0.13)
11	0.0654	0.0136	(9,-0.06)(12, 0.16)(93, 0.06)
12	0.4863	0.1022	(3,-0.14)(6, 0.10)(12, 0.07)
13	0.6188	0.1962	(1, 0.27)(7,-0.06)(11,-0.14)
14	0.5936	0.2635	(2, 0.36)(8,-0.09)(12,-0.25)
15	3.5281	0.8466	(21,-0.07)(27, 0.07)(33,-0.07)(39, 0.07) (45,-0.07)(51, 0.07)(57,-0.07)(63, 0.07) (69,-0.07)(75, 0.07)(81,-0.06)
16	3.4818	0.8272	(22,-0.06)(28, 0.07)(34,-0.07)(40, 0.07) (46,-0.07)(52, 0.07)(58,-0.07)(64, 0.07) (70,-0.07)(76, 0.07)(82,-0.06)

Computer Output No. 13. Brown's system observing all outputs
Time = 12:10 PM

VII. SUMMARY AND CONCLUSIONS

This thesis expanded the idea of Brown with regard to the question, "How observable?". The criterion for the measure of how observable the system is was more fully developed. This overall system criterion turned out to be the smallest eigenvalue and its associated eigenvector of the symmetric $Q_N Q_N^T$ matrix.

In addition, two more criteria were developed which are measures of how observable each state-variable of the system is. One of the criteria is based on the standard-deviation error analysis and the other is based on the upper-bound error analysis.

The numerical techniques for calculating these criteria were fully developed. Two inertial navigation systems were used as examples to test these criteria. The results are contained in this thesis.

A method was developed to compute the Q matrix of a time-varying system. It involved differentiating a function a considerable number of times. This differentiating was done on the computer algebraically rather than numerically.

By using the criteria developed in this research, a designer of a complex system should be able to gain a much better insight into his system with less calculation than by other methods available to him. Exactly how these criteria would be used would depend on the specifications of the system and the designer using them.

It should be pointed out that all criteria are obtained from the Q matrix and can be applied to the controllability Q matrix as well.

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X. APPENDIX A - FORTRAN PROGRAM FOR CALCULATING THE OBSERVABILITY CRITERIA

The program is as given in Computer Output Number 14. The subroutine for calculating the eigenvalues and eigenvectors is given at the end of the main program.

OS/360 FORTRAN H

```
C     COMPUTATION OF OBSERVABILITY FUNCTIONS
      DOUBLE PRECISION Q(18,96),U(18,18),A(324),R(324),F(18),
     1X(18),AM(18,18),CD(4),C(16,2,16),FN(4),CC(512),P(18,96),
     2CM(16,4),VT(1728),FI(5),T,PI,W,WT,SB,PI6,CTX,CX,TCX,
     3ANRMX,ANORM
      INTEGER IDA(4),IDS(4),IP(5),IPD(1728),IS(18)
      EQUIVALENCE (Q(1),VT(1)),(P(1),IPD(1)),(C(1),P(865)),
     1(C(1),CC(1)),(C(1),CM(1))
1     FORMAT (5I3)
2     FORMAT (/T2,10D12.4/(T5,10D12.4))
3     FORMAT (A1)
6     FORMAT (2I3,D16.7)
      READ (1,3) PL
C CLEAR A MATRIX AND INPUT NEW VALUES
22    READ (1,1) N
      IF (N) 24,24,25
25    CONTINUE
      NN=N*N
      DO 9 I=1,N
      DO 9 J=1,N
9     AM(I,J)=0.0D0
8     READ(1,6) I,J,T
      IF (I) 12,12,15
15    AM(I,J)=T
      GO TO 8
12    CONTINUE
C COMPUTE COMPONENTS OF EARTH'S ROTATION RATE AND INSERT INTO
C A MATRIX
      PI=3.14159265358979D0
      W=15.04107D0*PI/(180.0D0*3600.0D0)
      SB=1.0D0/DSQRT(2.0D0)
      PI6=PI/6.0D0
      AM(1,2)=W*DSQRT(1.0D0-SB*SB)
      AM(2,1)=(-AM(1,2))
      AM(2,3)=W*SB
      AM(3,2)=(-AM(2,3))
C OUTPUT A MATRIX
30    FORMAT ('1  A MATRIX')
      WRITE (3,30)
      DO 90 I=1,N
90    WRITE (3,2) (AM(I,J),J=1,N)
C CLEAR C MATRIX AND INPUT NEW VALUES
      READ (1,1) NM
      M=N*NM
      NQC=M
      DO 16 I=1,512
16    CC(I)=0.0D0
```

Computer Output No. 14. Fortran program for calculating the observability criteria

```
31     READ (1,6) I,J,T
       IF (I) 35,35,32
32     CM(I,J)=T
       GO TO 31
C OUTPUT C MATRIX
5      FORMAT (//T3,'C MATRIX')
35     WRITE (3,5)
       DO 17 I=1,NM
17     WRITE (3,2) (CM(I,J),J=1,N)
C FORM Q MATRIX
       DO 21 I=1,NM
       DO 21 J=1,N
21     Q(J,I)=CM(I,J)
       NT=NM+1
       DO 18 J=NT,M
       IL=J-NM
       DO 18 I=1,N
       Q(I,J)=0.0
       DO 18 K=1,N
18     Q(I,J)=Q(I,J) +Q(K,IL)*AM(K,I)
129    FORMAT (//T3,'Q MATRIX')
       WRITE (3,129)
       DO 125 I=1,N
125    WRITE (3,2) (Q(I,J),J=1,NQC)
202    CONTINUE
C NORMALIZE Q MATRIX
       DO 14 J=1,NQC
       T=0.0D0
       DO 10 I=1,N
10     T=T+(Q(I,J)*Q(I,J))
       IF (T) 14,14,19
19     T=1.0D0/DSQRT(T)
       DO 11 I=1,N
11     Q(I,J)=Q(I,J)*T
14     CONTINUE
C FORM PRODUCT OF NORMALIZED Q MATRIX AND ITS TRANSPOSE AND
C COMPUTE THESHOLD LEVEL FOR NEXT PART
       K=0
       ANORM=0.0D0
       DO 40 I=1,N
       X(I)=1.0D0
       DO 40 J=1,I
       K=K+1
       A(K)=0.0D0
       DO 46 L=1,M
46     A(K)=A(K)+Q(I,L)*Q(J,L)
       IF (I-J) 41,40,41
41     ANORM=ANORM+A(K)*A(K)
```

Computer Output No. 14 (Continued)

```
40     CONTINUE
C CALCULATE EIGENVALUES AND EIGENVECTORS BY JACOBI METHOD
       CALL EIGEN (A,R,N,F,ANORM,ANRMX)
C COMPUTE THESHOLD LEVEL AND SET TO ZERO ALL EIGENVALUES AND
C ELEMENTS OF EIGENVECTORS WHOSE ABSOLUTE VALUE IS LESS THAN
C THE THESHOLD LEVEL
       IF (ANRMX) 72,81,72
81     ANRMX=1.0D-12
72     ANRMX=ANRMX*1.0D+3
27     FORMAT (//T3,'THRESHOLD =',D14.7)
       WRITE (3,27) ANRMX
       DO 374 I=1,NN
       IF (DABS(R(I))-ANRMX) 366,366,374
366    R(I)=0.0D0
374    CONTINUE
       DO 65 I=1,N
       IF (F(I)-ANRMX) 73,73,74
73     F(I)=0.0D0
       A(I)=0.0D0
C CHECK, IF STATE VARIABLE IS NOT OBSERVABLE, SET INDICATOR
       K=(I-1)*N
       DO 66 J=1,N
       K=K+1
       IF (R(K)) 45,66,45
45     X(J)=0.0D0
66     CONTINUE
       GO TO 65
74     A(I)=F(I)
C INVERT DIAGONAL MATRIX
       F(I)=1.0D0/F(I)
65     CONTINUE
C OUTPUT OBSERVABILITY FUNCTIONS AND EIGENVECTORS
34     FORMAT ('1',/////,T14,'OBSERVABILITY FUNCTIONS')
       WRITE (3,34)
92     FORMAT (T10,4D15.4)
       WRITE (3,92) (A(J),J=1,N)
33     FORMAT (T14,'STATE VARIABLE NO. ON LEFT MARGIN',/T19,
      1'SIX SMALLEST EIGENVALUES WITH ASSOCIATED EIGENVECTORS')
       WRITE (3,33)
402    FORMAT (T16,6F10.5)
       WRITE (3,402) (A(J),J=1,6)
403    FORMAT ('+', T19,'_______   _______   _______',
      1'_______   _______   _______')
       WRITE (3,403)
       N6=N*6
       DO 365 I=1,N
204    FORMAT (T14,I2,6F10.5)
365    WRITE (3,204) I,(R(J),J=I,N6,N)
```

Computer Output No. 14 (Continued)

```
C CHECK, IF NO STATE VARIABLE IS OBSERVABLE, OUTPUT MESSAGE
C AND GO TO END
      T=0.0D0
      DO 57 I=1,N
57    T=T+X(I)
      IF (T) 69,68,69
94    FORMAT (//T14,'NONE OF THE STATE VARIABLES ARE ',
     1'OBSERVABLE')
68    WRITE (3,94)
      GO TO 79
C COMPUTE GENERALIZED INVERSE
69    DO 76 I=1,N
      DO 76 J=1,N
      L1=I
      L2=J
      U(I,J)=0.0D0
      DO 76 K=1,N
      U(I,J)=U(I,J)+R(L1)*E(K)*R(L2)
      L1=L1+N
76    L2=L2+N
      DO 80 J=1,M
      DO 77 I=1,N
      A(I)=0.0D0
      DO 77 K=1,N
77    A(I)=A(I)+U(I,K)*Q(K,J)
      DO 80 I=1,N
80    P(I,J)=A(I)
C OUTPUT HEADINGS
60    FORMAT (T14,'DEGREE OF OBSERVABILITY PER STATE VARI',
     1'ABLE',/T17,'STANDARD   UPPER   OBSERVED OUTPUT NO. ',
     2'AND PROPORTIONAL',/T17,'DEVIATION  BOUND    PART OF ',
     3'THAT VALUE IF PART IS OVER 0.1')
      WRITE (3,60)
C FOR EACH STATE VARIABLE , DO THE FOLLOWING
      DO 61 I=1,N
C CHECK, IF STATE VARIABLE IS NOT OBSERVABLE, OUTPUT MESSAGE
C AND GO TO NEXT STATE VARIABLE
      IF (X(I)) 84,85,84
93    FORMAT (T14,I2,T17,'NOT OBSERVABLE')
85    WRITE (3,93) I
      GO TO 61
C CALCULATE STANDARD DEVIATION OBSERVABILITY
84    T=0.0D0
      DO 62 J=1,M
62    T=T+P(I,J)*P(I,J)
```

Computer Output No. 14 (Continued)

```
C CHECK, IF STANDARD DEVIATION OBSERVABILITY IS ZERO, OUTPUT
C ZERO FOR BOTH TYPES OF OBSERVABILITIES AND GO TO NEXT
C STATE VARIABLE
      IF (T) 309,309,63
309   ANCRM=T
      WRITE (3,7) I,ANCRM,T
      GO TO 61
63    ANORM=1.0D0/DSQRT(T)
C CALCULATE UPPER BOUND OBSERVABILITY
86    T=0.0D0
      DO 38 J=1,M
38    T=DABS(P(I,J))+T
52    T=1.0D0/T
C COMPUTE DECIMAL PART OF EACH OBSERVED OUTPUT VALUE IN THE
C STATE VARIABLE
      ISS=0
319   DO 320 J=1,M
      A(J)=P(I,J)*T
C CHECK, IF DECIMAL PART IS OVER 0.1, STORE, TO BE USED LATER
      IF (DABS(A(J))-0.1D0) 320,320,51
51    ISS=ISS+1
      A(ISS)=A(J)
      IS(ISS)=J
320   CONTINUE
C OUTPUT DEGREE OF OBSERVABILITIES AND DECIMAL PARTS OVER 0.1
7     FORMAT (T14,I2,T17,F7.4,T27,F7.4,(T36,4(A1,I2,',',F5.2,
     1')')))
      WRITE (3,7) I,ANCRM,T,(PL,IS(J),A(J),J=1,ISS)
61    CONTINUE
79    CONTINUE
300   CONTINUE
      GO TO 22
23    FORMAT ('1',//T3,'END OF PROBLEMS')
24    WRITE (3,23)
      STOP
      END
```

Computer Output No. 14 (Continued)

```
      SUBROUTINE EIGEN (A,R,N,F,ANORM,ANRMX)
      DIMENSION A(1),R(1),F(1)
      DOUBLE PRECISION A,R,ANORM,ANRMX,THR,X,Y,SINX,SINX2,
     1           COSX,COSX2,SINCS,F
      NN=N*N
      J=N+1
      DO 220 I=1,NN
220   R(I)=0.0
      DO 215 I=1,NN,J
215   R(I)=1.0
C        COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)
      IF(ANORM) 165,165,40
   40 ANORM=1.414*DSQRT(ANORM)
      ANRMX=ANORM*1.0D-12/FLOAT(N)
C        INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR
      IND=0
      THR=ANORM
   45 THR=THR/FLOAT(N)
   50 L=1
   55 M=L+1
C        COMPUTE SIN AND COS
   60 MQ=(M*M-M)/2
      LQ=(L*L-L)/2
      LM=L+MQ
   62 IF(DABS(A(LM))-THR) 130,65,65
   65 IND=1
      LL=L+LQ
      MM=M+MQ
      X=0.5*(A(LL)-A(MM))
   68 Y=-A(LM)/DSQRT(A(LM)*A(LM)+X*X)
      IF(X) 70,75,75
   70 Y=-Y
   75 SINX=Y/DSQRT(2.0*(1.0+(DSQRT(1.0-Y*Y))))
      SINX2=SINX*SINX
   78 COSX=DSQRT(1.0-SINX2)
      COSX2=COSX*COSX
      SINCS =SINX*COSX
C        ROTATE L AND M COLUMNS
      ILQ=N*(L-1)
      IMQ=N*(M-1)
      DO 125 I=1,N
      IQ=(I*I-I)/2
      IF (I-L) 80,120,80
80    IF (I-M) 85,120,90
   85 IM=I+MQ
      GO TO 95
   90 IM=M+IQ
```

Computer Output No. 14 (Continued)

```
   95 IF(I-L) 100,105,105
  100 IL=I+LQ
      GO TO 110
  105 IL=L+IQ
  110 X=A(IL)*COSX-A(IM)*SINX
      A(IM)=A(IL)*SINX+A(IM)*COSX
      A(IL)=X
  120 ILR=ILQ+I
      IMR=IMQ+I
      X=R(ILR)*COSX-R(IMR)*SINX
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX
      R(ILR)=X
  125 CONTINUE
      X=2.0*A(LM)*SINCS
      Y=A(LL)*COSX2+A(MM)*SINX2-X
      X=A(LL)*SINX2+A(MM)*COSX2+X
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)
      A(LL)=Y
      A(MM)=X
C         TESTS FOR COMPLETION
C         TEST FOR M = LAST COLUMN
  130 IF(M-N) 135,140,135
  135 M=M+1
      GO TO 60
C         TEST FOR L = SECOND FROM LAST COLUMN
  140 IF(L-(N-1)) 145,150,145
  145 L=L+1
      GO TO 55
  150 IF(IND-1) 160,155,160
  155 IND=0
      GO TO 50
C         COMPARE THRESHOLD WITH FINAL NORM
  160 IF(THR-ANRMX) 165,165,45
C         SORT EIGENVALUES AND EIGENVECTORS
165   DO 170 J=1,N
      K=((J+1)*J)/2
170   F(J)=A(K)
      DO 185 I=1,N
      X=F(I)
      DO 172 J=I,N
      IF (X-F(J)) 172,173,171
171   X=F(J)
173   L=J
172   CONTINUE
      IF (L-I) 185,185,174
174   F(L)=F(I)
      F(I)=X
      IM=(I-1)*N
```

Computer Output No. 14 (Continued)

```
      IL=(L-1)*N
      DO 180 K=1,N
      IM=IM+1
      IL=IL+1
      X=R(IL)
      R(IL)=R(IM)
180   R(IM)=X
  185 CONTINUE
      RETURN
      END
```

Computer Output No. 14 (Continued)

XI. APPENDIX B - FORTRAN PROGRAM FOR CALCULATING THE GENERALIZED INVERSE

The computer program given in Computer Output Number 15 is a slightly modified version of the program due to Rust, Burrus, and Schneeberger (30).

```
      SUBROUTINE PINI (A,U,F,M,N)
      DIMENSION A(80,20),U(20,20),F(20),T(20)
      DOUBLE PRECISION A,U,F,T,D1,D2,TOL
      DO 10 I=1,N
      DO 5 J=1,N
5     U(I,J)=0.0
      U(I,I)=1.0
10    F(I)=0.0
      TOL=1.0D-20
      DO 100 J=1,N
      D2=0.0
      DO 7 I=1,M
7     D2=D2+A(I,J)*A(I,J)
      IF (D2) 100,100,12
12    JM=J-1
      IF (JM) 70,70,8
8     DO 50 L=1,2
      DO 30 K=1,JM
      T(K)=0.0
      DO 30 I=1,M
30    T(K)=T(K)+A(I,J)*A(I,K)
      DO 45 K=1,JM
      IF (F(K)) 36,36,34
34    DO 35 I=1,M
35    A(I,J)=A(I,J)-T(K)*A(I,K)
36    DO 40 I=1,K
40    U(I,J)=U(I,J)-T(K)*U(I,K)
45    CONTINUE
50    CONTINUE
      D1=D2
      D2=0.0
      DO 11 I=1,M
11    D2=D2+A(I,J)*A(I,J)
      IF ((D2/D1)-TOL) 55,55,70
55    DO 60 I=1,JM
      T(I)=0.0
      DO 60 K=1,I
60    T(I)=T(I)+U(K,I)*U(K,J)
      DO 65 I=1,M
      A(I,J)=0.0
      DO 65 K=1,JM
65    A(I,J)=A(I,J)-A(I,K)*T(K)*F(K)
      D2=0.0
      DO 16 I=1,J
16    D2=D2+U(I,J)*U(I,J)
      GO TO 75
```

Computer Output No. 15. Fortran program for calculating the generalized inverse

```
70     F(J)=1.0
75     D2=1.0/DSQRT(D2)
       DO 80 I=1,M
80     A(I,J)=A(I,J)*D2
       DO 85 I=1,J
85     U(I,J)=U(I,J)*D2
100    CONTINUE
       DO 130 J=1,N
       DO 130 I=1,M
       D2=0.0
       DO 120 K=J,N
120    D2=D2+A(I,K)*U(J,K)
130    A(I,J)=D2
       RETURN
       END
```

Computer Output No. 15 (Continued)

XII. APPENDIX C - FORMULATION OF THE Q MATRIX FOR THE LINEAR TIME VARYING SYSTEM

For the 16 state-variable system, some of the elements of the C matrix are time varying as shown in Table 2. The variable elements are defined as shown in Equation A1 through A6.

$$C_{\mu x} = \frac{\cos \Omega t}{\sqrt{1 - S_b^2 \sin^2 \Omega t}} \tag{A1}$$

$$C_{\mu y} = \frac{-\sqrt{1 - \cos^2 \Omega t - S_b^2 \sin \Omega t}}{\sqrt{1 - S_b^2 \sin^2 \Omega t}} \tag{A2}$$

$$C_{\mu z} = 0 \tag{A3}$$

$$C_{vx} = \frac{S_b \sin \Omega t \sqrt{1 - \cos^2 \Omega t - S_b^2 \sin^2 \Omega t}}{\sqrt{1 - S_b^2 \sin^2 \Omega t}} \tag{A4}$$

$$C_{vy} = \frac{S_b \sin \Omega t \cos \Omega t}{\sqrt{1 - S_b^2 \sin^2 \Omega t}} \tag{A5}$$

$$C_{vz} = \sqrt{1 - S_b^2 \sin^2 \Omega t} \tag{A6}$$

S_b is a constant depending on the latitude and Ω is the earth's rotational rate in rad./sec. The unit of time used is seconds. For a more detailed information on these elements, see the paper by Brown and Friest (8).

According to Equation 7, fifteen derivatives must be taken to form the full Q matrix for this sixteen state-variable system. Instead of forming the Q matrix as shown in Equation 7, the C matrix was differentiated and substituted into the P matrices as shown by the set of equations numbered A7.

$$
\begin{aligned}
P_1 &= C^T \\
P_2 &= A^T C^T + \dot{C}^T \\
P_3 &= A^{T^2} C^T + 2A^T \dot{C}^T + \ddot{C}^T \\
P_4 &= A^{T^3} C^T + 3(A^T)^2 \dot{C}^T + \dddot{C}^T \\
P_5 &= A^{T^4} C^T + 4(A^T)^3 C^T + 6(A^T)^2 \ddot{C}^T + 4A^T \dddot{C}^T + \ddddot{C}^T
\end{aligned}
\tag{A7}
$$

The number of dots above the symbols indicates how many times the matrix has been differentiated with respect to time. The coefficients of matrices are the binomial coefficients.

To differentiate the C matrix, each element of the C matrix was differentiated as many times as required and the value substituted into the C matrix.

Four parameters were chosen so that when they were differentiated with respect to time, the differentiated term was a constant times a product of the four parameters. The parameters chosen are shown in the Equations A8 through A11.

$$w = \sqrt{1 - \cos^2\Omega t - S_b{}^2 \sin \Omega t} \tag{A8}$$

$$x = \cos \Omega t \tag{A9}$$

$$y = S_b \sin \Omega t \tag{A10}$$

$$z = \frac{1}{\sqrt{1 - S_b{}^2 \sin^2\Omega t}} \tag{A11}$$

The derivative with respect to time is given in Equations A12 through A15.

$$\frac{dw}{d(\Omega t)} = [\frac{1}{S_b} - S_b] \, xyw^{-1} \tag{A12}$$

$$\frac{dx}{d(\Omega t)} = - \frac{1}{S_b} \, y \tag{A13}$$

$$\frac{dy}{d(\Omega t)} = S_b \, x \tag{A14}$$

$$\frac{dz}{d(\Omega t)} = S_b \, xyz^3 \tag{A15}$$

The elements of the C matrix are given as functions of the four parameters in Equation A16 through A20.

$$C_{\mu x} = xz \tag{A16}$$

$$C_{\mu y} = -wz \tag{A17}$$

$$C_{vx} = wyz \tag{A18}$$

$$C_{vy} = xyz \tag{A19}$$

$$C_{vz} = z^{-1} \tag{A20}$$

Differentiating Equation A16 by the chain rule with respect to Ωt results in Equation A21.

$$\dot{C}_{\mu x} = -(\frac{1}{S_b}) \, yz + S_b \, x^2yz^3 \tag{A21}$$

The first term of Equation A21 can be obtained by multiplying the $C_{\mu x}$ term by $- \frac{1}{S_b} x^{-1}y$ and the second term by multiplying by $S_b \, xyz^2$. A set of multiplying terms were formed for the parameters as shown in Equations A22 through A25.

$$wM = [\frac{1}{S_b} - S_b]\, w^{-2}\, xy \qquad (A22)$$

$$xM = -(\frac{1}{S_b})\, x^{-1}y \qquad (A23)$$

$$yM = S_b\, xy^{-1} \qquad (A24)$$

$$zM = S_b\, xyz^2 \qquad (A25)$$

For each term two positions in memory are needed; one to keep track of the exponents and the other to carry the value of the term. After the initial value and exponent has been entered into the memory for a function, a search is made for the first non zero exponent of the first term. When it is found, the exponents are added to the exponents of the multiplier term and the value of the term is multiplied by the value of the multiplier value. The new pair is stored in another place in memory reserved for the derivative. The value of each term is added to the memory position which contains the value of the derivative.

Each time a new term is formed a search is made through all the other terms of the derivative to find another term with the same set of exponents. If another term is found, the two are combined to form one term. If the value of the new term is zero the term is eliminated completely. This procedure is followed because of the increasing number of terms with each differentiation. For example, if we start with a term with three parameters and assume that all terms after the first differentiation will contain all four terms, not combining the term would result in about 800 million terms on the 15th differentiation. With the combination and elimination, the 15th differentiation may contain about 1000 terms.

The number of terms in the 14th differentiation was counted by the computer. Without the combination and elimination of terms, an estimated 200 million terms could result. With the combination and elimination of terms, the 14th differentiation had 64 terms for $C_{\mu x}$, 511 terms for $C_{\mu y}$, 512 terms for C_{vx}, 64 terms for C_{vy}, and 63 terms for C_{vz}. The terms for the 14th differentiation were counted to insure that enough memory space was allotted in the computer program.

The Fortran program is given in Computer Output Number 16. More details about this method can be obtained from the program. This program was inserted in the program given in Computer Output Number 14 replacing the part of the program which formed the C and Q matrix.

```
C SET NUMBER OF SYSTEM OUTPUTS AND DERIVATIVES TO BE TAKEN
      NM=6
      ND=N-1
      ND1=ND+1
      M=NM*ND1
      NCC=M
C SET ADDITIVE DERIVATIVE PARAMETERS POWERS
      IDA(1)=257*256
      IDS(1)=2*256*256*256
      IDA(2)=256
      IDS(2)=256*256
      IDA(3)=256*256
      IDS(3)=256
      IDA(4)=257*256+2
      IDS(4)=0
      DO 142 I=1,4
142   IDA(I)=IDA(I)-IDS(I)
C SET FUNCTION PARAMETER POWERS
      IP(1)    =((64*256+65)*256+64)*256+65
      IP(2)    =((65*256+64)*256+64)*256+65
      IP(3)    =((65*256+64)*256+65)*256+65
      IP(4)    =((64*256+65)*256+65)*256+65
      IP(5)    =((64*256+64)*256+64)*256+63
      ICO=5
      DO 300 ITOTAL=1,4
      WT=(ITOTAL-1)*PI6
      ITEST=1
      ITEN=1
      ITIME=4 +2*ITOTAL
      IF (ITIME-12) 147,146,145
145   ITIME=ITIME-12
146   ITEST=ITEST+1
C CHECK, IF ANY PARAMETERS ARE ZERO, ADD 10 MIN. TO TIME
147   FN(3)=DSIN(WT)
      IF (DABS(FN(3))-(1.0D-16)) 108,108,107
107   FN(2)=DSQRT(1.0D0-FN(3)*FN(3))
      IF (DABS(FN(2))-(1.0D-16)) 108,108,155
108   WT=WT+PI6/12.0D0
      ITEN=2
```

Computer Output No. 16. Fortran program for formulation of the Q matrix for the linear time varying system

```
C INSERT VALUE OF PARAMETERS
      FN(3)=DSIN(WT)
      FN(2)=DSQRT(1.0D0-FN(3)*FN(3))
155   FN(3)=SB*FN(3)
      FN(1)=DSQRT(1.0D0-FN(2)*FN(2)-FN(3)*FN(3))
      FN(4)=1.0D0/DSQRT(1.0D0-FN(3)*FN(3))
C INSERT VALUE OF FUNCTIONS
      FI(1)  =FN(2)*FN(4)
      FI(2)  =(-FN(1))*FN(4)
      FI(3)  =FN(1)*FN(3)*FN(4)
      FI(4)  =FN(2)*FN(3)*FN(4)
      FI(5) =1.0D0/FN(4)
      DO 16 I=1,512
16    CC(I)=0.0D0
      DO 164 I=1,5
      IR=(3+I)/3
      IC=I-2*(IR-1)
164   C(1,IR,IC)=FI(I)
C INSERT VALUE OF DERIVATIVE MULIPLIERS
      CD(1)=  (1.0D0/SB-SB)*FN(2)*FN(3)/(FN(1)*FN(1))
      CD(2)=  FN(3)/(FN(2)*(-SB))
      CD(3)=  SB*FN(2)/FN(3)
      CD(4)=  SB*FN(2)*FN(3)*FN(4)*FN(4)
127   CONTINUE
C OUTPUT VALUE OF PARAMETERS AND DERIVATIVE MULTIPLIERS
270   FORMAT('1',T3,'VALUES OF THE PARAMETERS')
      WRITE (3,270)
203   FORMAT (/T3,4('FN(',I2,')=',F12.9,'    '))
      WRITE(3,203) (J,FN(J),J=1,4)
273   FORMAT (//T3,'VALUES OF THE DERIVATIVE MULTIPLIERS')
274   WRITE (3,273)
205   FORMAT  (/T3,4('CD(',I2,')=',F12.9,'    '))
275   WRITE (3,205) (J,CD(J),J=1,4)
C REPEAT THE FOLLOWING TO 117 FOR EACH FUNCTION
      DO 117 J=1,5
C CLEAR AND INITIALIZE WORKING MEMORY
106   DO 141 I=1,1728
      VT(I)=0.0D0
141   IPD(I)=0
      VT(1)=FI(J)
      IPD(1)=IP(J)
C SET INDEX VALUES AND COUNTERS
      IR=(3+J)/3
      IC=J-2*(IR-1)
      KS=1
      KP=1
```

Computer Output No. 16 (Continued)

```
C REPEAT THE FOLLOWING TO 117 FOR EACH DERIVATIVE TO BE TAKEN
      DO 117 I=1,ND
C SET COUNTERS SO THAT WORKING MEMORY CAN BE FILLED FROM
C ALTERNATE ENDS FOR EACH SUCCESSIVE DERIVATIVE
      KS=KS*(-1)
      IF (KS) 150,150,151
150   K=1728
      L=KP
      GO TO 152
151   K=1
      L=1729-KP
152   LB=KP
      KP=0
C REPEAT TO 116 FOR EACH TERM
      DO 116 LA=1,LB
      IFX1=IPD(L)
      CTX=VT(L)
C REPEAT TO 103 FOR EACH PARAMETER
      DO 103 JA=1,4
      JB=5-JA
C EXTRACT THE POWER OF THE PARAMETER
      IFX2=IFX1
      IFX1=IFX1/256
      IFX2=IFX2-IFX1*256-64
C CHECK, IF POWER IS ZERO, GO TO NEXT PARAMETER
      IF (IFX2) 110,103,110
C DIFFERENTIATE WITH RESPECT TO PARAMETER BY ADDING ADDITIVE
C DERIVATIVE PARAMETER POWERS TO PARAMETER POWER OF TERM AND
C BY MULITPLYING DERIVATIVE MULTIPLIER BY VALUE OF TERM AND
C POWER OF PARAMETER
110   IX1=IPD(L)+IDA(JB)
      CX=CTX*CD(JB)*IFX2
C CHECK, IF THIS IS THE FIRST TERM OR THE LAST DERIVATIVE,
C ELIMINATE THE FOLLOWING CHECKS
      IF (KP) 102,210,102
102   IF (ND-I) 162,162,101
C CHECK, IF NEWEST TERM HAS SAME PARAMETER POWERS AS ANY OTHER
C TERM, ADD THE VALUES OF THE TWO TERMS
101   LK=K-KS
      DO 181 LTT=1,KP
      IF (IX1-IPD(LK)) 181,182,181
C CHECK, IF THE VALUE OF THE SUM OF THE TWO TERMS IS LESS THAN
C THE THESHOLD, REDUCE TERM COUNTER BY ONE AND STORE LAST
C PREVIOUS TERM IN THAT POSITION
182   ANRMX=(DABS(CX-VT(LK)))*(1.0D-14)*I
      TCX=CX+VT(LK)
```

Computer Output No. 16 (Continued)

```
      IF (DABS(TCX)-ANRMX) 184,184,183
183   VT(LK)=TCX
      GO TO 162
184   KP=KP-1
      K=K-KS
      IPD(LK)=IPD(K)
      VT(LK)=VT(K)
      GO TO 162
181   LK=LK-KS
C STORE VALUE AND PARAMETER POWER OF TERM IN NEW POSITION
C ADD VALUE OF TERM TO VALUE OF DERIVATIVE OF FUNCTION
210   IPD(K)=IX1
      VT(K)=CX
185   K=K+KS
      KP=KP+1
162   C(I+1,IR,IC)=C(I+1,IR,IC)+CX
103   CONTINUE
116   L=L+KS
C OUTPUT THE NUMBER OF TERMS IN THE NEXT TO THE LAST
C DERIVATIVE, AND THE VALUE OF EACH DERIVATIVE
      IF (I+1-ND) 117,211,117
153   FORMAT (/T3,'DERIV. NO.=',I2,'    FUNC. NO.=',I2,'    TERM
211   WRITE (3,153) I,J,KP                                  1S=',I4)
117   CONTINUE
      DO 216 I=1,ND1
112   FORMAT (/T3,'DERIVATIVE NO.=',I2)
      ID=I-1
      WRITE (3,112) ID
      DO 216 J=1,2
216   WRITE (3,5) (C(I,J,K),K=1,3)
C CLEAR Q MATRIX AND INSERT C MATRIX INTO FIRST 6 COLUMNS
      DO 140 I=1,1728
140   VT(I)=0.0D0
      Q(5,1)=1.0D0
      Q(13,1)=1.0D0
      Q(7,2)=1.0D0
      Q(14,2)=1.0D0
      Q(15,3)=1.0D0
      Q(16,4)=1.0D0
      Q(4,5)=1.0D0
      Q(6,6)=1.0D0
      Q(1,3)=C(1,1,1)
      Q(2,3)=C(1,1,2)
      Q(1,4)=C(1,2,1)
      Q(2,4)=C(1,2,2)
      Q(3,4)=C(1,2,3)
      DO 130 L=1,ND
      IF (L-1) 122,122,124
```

Computer Output No. 16 (Continued)

```
C MULTIPLY THE DERIVATIVES BY THE A MATRIX
124     DO 126 LD=2,L
        DO 126 J=1,2
        DO 128 I=1,N
        X(I)=0.0D0
        DO 128 K=1,N
128     X(I)=X(I)+AM(K,I)*C(LD,J,K)
        DO 126 I=1,N
126     C(LD,J,I)=X(I)
C MULTIPLY THE NEXT 6 COULMNS BY THE A MATRIX
122     DO 131 J=1,NM
        I2=L*NM +J
        I1=I2-NM
        DO 131 I=1,N
        DO 131 K=1,N
131     Q(I,I2)=Q(I,I2)+Q(K,I1)*AM(K,I)
C TO THE Q MATRIX, ADD THE PRODUCT OF THE DERIVATIVE AND ITS
C PROPER COEFFICIENT
133     IC=1
        LX=L+1
        DO 134 IX=2,LX
        CX=FLOAT(IC)
        I1=L*NM +2
        DO 132 J=1,2
        I1=I1+1
        DO 132 K=1,N
132     Q(K,I1)=Q(K,I1)+C(IX,J,K)*CX
134     IC=(IC*(LX-IX))/(IX-1)
130     CONTINUE
202     CONTINUE
```

Computer Output No. 16 (Continued)